

Economic Dispatch

- **In practice and in power flow analysis, there are many choices for setting the operating points of generators**
 - ◆ in the power flow analysis, generator buses are specified by P and $|V|$
 - ◆ generation capacity is more than load demand - generators can produce more than the customers can consume
 - there are many solution combinations for scheduling generation
 - ◆ in practice, power plants are not located at the same distance from the load centers
 - ◆ power plants use different types of fuel, which vary in cost from time to time
- **For interconnected systems, the objective is to find the real and reactive power scheduling so as to minimize some operating cost or cost function**

Power Systems I

Optimization

- **General cost function:** $f(x_1, x_2, \dots, x_n) = C$
- **Unconstrained parameter optimization, from calculus:**

- ◆ the first derivative of f vanishes at a local extrema

$$\frac{d}{dx} f(x) = 0$$

- ◆ for f to be a local minimum, the second derivative must be positive at the point of the local extrema

$$\frac{d^2}{dx^2} f(x) > 0$$

- ◆ for a set of parameters, the gradient of f vanishes at a local extrema and to be a local minimum, the Hessian must be a positive definite matrix (i.e. positive eigenvalues)

$$\frac{\partial f}{\partial x_i} = 0 \quad i = 1, \dots, n \quad \text{or} \quad \nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right) = 0$$

Optimization

- The Hessian matrix

$$H_{ij} = \frac{\partial^2 f(\hat{x}_1 \cdots \hat{x}_i \cdots \hat{x}_n)}{\partial x_i \partial x_j}$$

- ◆ a symmetrical matrix
- ◆ contains the second derivatives of the function f
- ◆ for f to be a minimum, the Hessian matrix must be positive definite

$$\mathbf{x}^T \mathbf{H} \mathbf{x} > 0 \quad \forall \mathbf{x} \neq \mathbf{0}$$

- ◆ this condition also requires that all the eigenvalues of the Hessian matrix evaluated at the extrema to be positive

$$0 < \text{eigen}_i [\mathbf{H}(\hat{x}_1 \cdots \hat{x}_j \cdots \hat{x}_n)] \quad i = 1, \dots, n$$

Example

- Find the minimum of

$$f(x_1, x_2, x_3) = x_1^2 + 2x_2^2 + 3x_3^2 + x_1x_2 + x_2x_3 - 8x_1 - 16x_2 - 32x_3 + 110$$

- ◆ evaluating the first derivatives to zero results in

$$\frac{\partial f}{\partial x_1} = 2x_1 + x_2 - 8 = 0$$

$$\frac{\partial f}{\partial x_2} = x_1 + 4x_2 + x_3 - 16 = 0 \quad \text{or}$$

$$\frac{\partial f}{\partial x_3} = x_2 + 6x_3 - 32 = 0$$

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 6 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 16 \\ 32 \end{bmatrix} \quad \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$$

Example

- ◆ evaluating the second derivatives and forming the Hessian matrix

$$H(\hat{x}) = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 6 \end{bmatrix}$$

- ◆ using the MATLAB function **eig(H)**, the eigenvalues are found

$$\text{eigen} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 6 \end{bmatrix} = \begin{bmatrix} 1.55 \\ 4.0 \\ 6.45 \end{bmatrix} > \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- ◆ the eigenvalues are all greater than zero, so it's a minimum point

Equality Constraints in Optimization

- This type of problem arises when there are functional dependencies among the parameters to be found
- The problem
 - ♦ minimize the cost function
$$f(\hat{x}_1 \cdots \hat{x}_j \cdots \hat{x}_n)$$
 - ♦ subject to the equality constraints
$$g_i(\hat{x}_1 \cdots \hat{x}_j \cdots \hat{x}_n) = 0 \quad i = 1, \dots, k$$
- Such problems may be solved by the *Lagrange multiplier* method

Equality Constraints in Optimization

- **Lagrange Multiplier method**

- ◆ introduce k -dimensional vector λ for the undetermined quantities

$$L = f + \sum_{i=1}^k \lambda_i g_i \quad \text{New cost function}$$

- ◆ The necessary conditions for finding the local minimum

$$\frac{\partial L}{\partial x_i} = \frac{\partial f}{\partial x_i} + \sum_{i=1}^k \lambda_i \frac{\partial g_i}{\partial x_i} = 0$$

$$\frac{\partial L}{\partial \lambda_i} = g_i = 0$$

Example

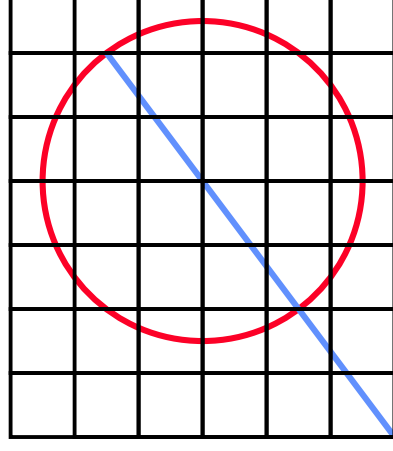
- Use the Lagrange multiplier method to determine the minimum distance from the origin of the x-y plane to a circle described by

$$(x-8)^2 + (y-6)^2 = 25 \quad \text{or}$$

$$g(x, y) = (x-8)^2 + (y-6)^2 - 25$$

- ♦ The minimum distance is obtained by minimizing the distance squared

$$f(x, y) = x^2 + y^2$$



Example

$$f(x, y) = x^2 + y^2 \quad g(x, y) = (x-8)^2 + (y-6)^2 - 25 = 0$$

$$L = f + \lambda \cdot g = x^2 + y^2 + \lambda[(x-8)^2 + (y-6)^2 - 25]$$

$$\frac{\partial L}{\partial x} = 2x + \lambda(2x-16) = 0 \quad \text{or} \quad 2x(\lambda+1) = 16\lambda$$

$$\frac{\partial L}{\partial y} = 2y + \lambda(2y-12) = 0 \quad \text{or} \quad 2y(\lambda+1) = 12\lambda$$

$$\frac{\partial L}{\partial \lambda} = (x-8)^2 + (y-6)^2 - 25 = 0$$

Example

- ♦ eliminating λ from the first two equations

$$\frac{16\lambda}{2x} = \frac{12\lambda}{2y} \rightarrow y = \frac{3}{4}x$$

- ♦ substituting for y in the third equation yields

$$(x-8)^2 + \left(\frac{3}{4}x-6\right)^2 - 25 = 0$$

$$\frac{25}{16}x^2 - 25x + 75 = 0 \rightarrow x = 4 \quad \& \quad x = 12$$

$$\text{extrema: } (4,3), \lambda = 1 \quad \text{and} \quad (12,9), \lambda = -3$$

$$x = 4$$

$$\min \rightarrow$$

$$y = 3$$

Power Systems I

Iterative Techniques

- **In many problems, a direct solution using Lagrange multiplier method is not possible**

- ♦ The equations are solved iteratively
- ♦ Newton-Raphson method is superior
- ♦ One possible way of casting the last example into an iterative process:

- rewrite the first two equations in terms of λ

$$x = \frac{8\lambda}{\lambda + 1} \quad y = \frac{6\lambda}{\lambda + 1} \quad f(\lambda) = 100 \left(\frac{\lambda}{\lambda + 1} \right)^2 + 200 \frac{\lambda}{\lambda + 1} + 75 = 0$$

- substitute the first two equations into the third equation
- the third equation is non-linear and in terms of a single variable, λ

$$\Delta \lambda^{[k]} = \frac{-\Delta f(\lambda^{[k]})}{\left(\partial f / \partial \lambda \right)^{[k]}} \quad \lambda^{[k+1]} = \lambda^{[k]} + \Delta \lambda^{[k]}$$

Power Systems I

Iterative Techniques

- ◆ Starting with an estimated value of λ , a new value is found in the direction of steepest descent
- ◆ The process is repeated until the error, $\Delta f(\lambda)$ is less than a specified accuracy
- ◆ This algorithm is known as the gradient method
- ◆ Numerical results of previous example, starting with a initial value $\lambda = 0.4$

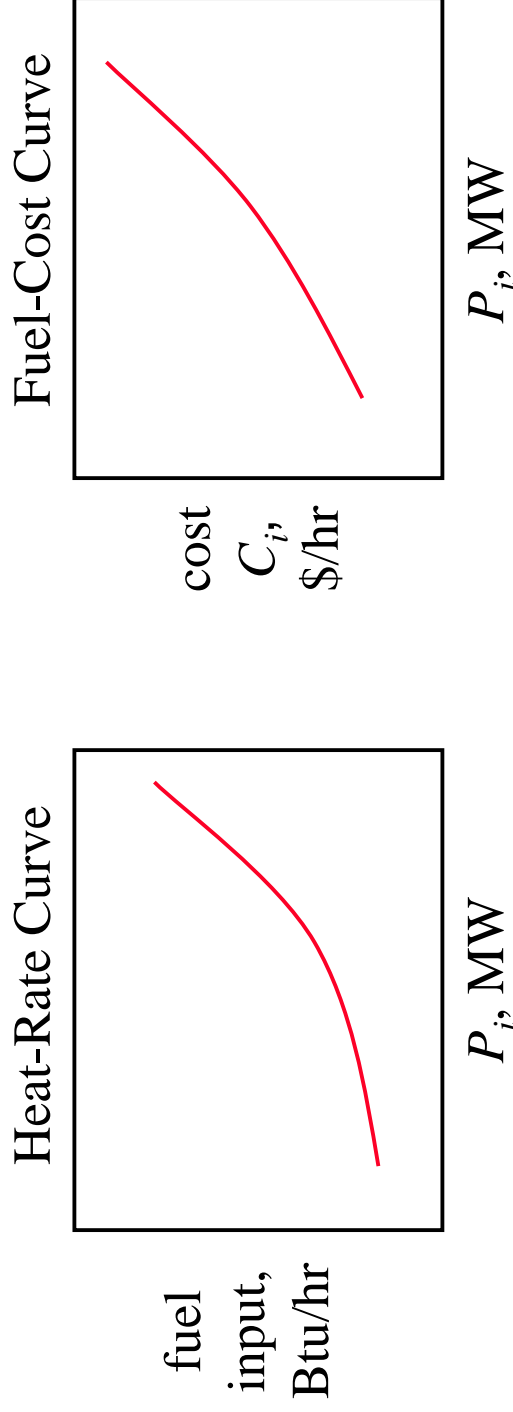
iter	Δf	J	$\Delta\lambda$	λ	x	y
1	26.02	-72.89	0.357	0.400	2.286	1.713
2	7.393	-36.87	0.201	0.757	3.447	2.585
3	1.097	-26.66	0.041	0.958	3.913	2.935
4	0.034	-25.05	0.001	0.999	3.997	2.998
5	0.000	-25.00	0.000	1.000	4.000	3.000

Operating Costs

- **Factors influencing the minimum cost of power generation**
 - ◆ operating efficiency of prime mover and generator
 - ◆ fuel costs
 - ◆ transmission losses
- **The most efficient generator in the system does not guarantee minimum costs**
 - ◆ may be located in an area with high fuel costs
 - ◆ may be located far from the load centers and transmission losses are high
- **The problem is to determine generation at different plants to minimize the total operating costs**

Operating Costs

- Generator heat rate curves lead to the fuel cost curves



- ◆ The fuel cost is commonly expressed as a quadratic function
- ◆ The derivative is known as the incremental fuel cost

$$C_i = \alpha_i + \beta_i P_i + \gamma_i P_i^2$$

$$\frac{dC_i}{dP_i} = \beta_i + 2\gamma_i P_i$$

Power Systems I

Economic Dispatch

- The simplest problem is when system losses and generator limits are neglected

- ◆ minimize the objective or cost function over all plants
- ◆ a quadratic cost function is used for each plant

$$C_{total} = \sum_{i=1}^{n_{gen}} C_i = \sum_{i=1}^{n_{gen}} \alpha_i + \beta_i P_i + \gamma_i P_i^2$$

- ◆ the total demand is equal to the sum of the generators' output; the equality constraint

$$\sum_{i=1}^{n_{gen}} P_i = P_{Demand}$$

Economic Dispatch

- A typical approach using the Lagrange multipliers

$$L = C_{total} + \lambda \left(P_{Demand} - \sum_{i=1}^{n_{gen}} P_i \right)$$

$$\frac{\partial L}{\partial P_i} = \frac{\partial C_{total}}{\partial P_i} + \lambda(0-1) = 0 \quad \rightarrow \quad \frac{\partial C_{total}}{\partial P_i} = \lambda$$

$$C_{total} = \sum_{i=1}^{n_{gen}} C_i \quad \rightarrow \quad \frac{\partial C_{total}}{\partial P_i} = \frac{dC_i}{dP_i} = \lambda \quad \forall i = 1, \dots, n_g$$

$$\lambda = \frac{dC_i}{dP_i} = \beta_i + 2\gamma_i P_i$$

Power Systems I

Economic Dispatch

- ◆ the second condition for optimal dispatch

$$\frac{dL}{d\lambda} = \left(P_{Demand} - \sum_{i=1}^{n_{gen}} P_i \right) = 0 \rightarrow \sum_{i=1}^{n_{gen}} P_i = P_{Demand}$$

- ◆ rearranging and combining the equations to solve for λ

$$P_i = \frac{\lambda - \beta_i}{2\gamma_i}$$

$$\sum_{i=1}^{n_{gen}} \frac{\lambda - \beta_i}{2\gamma_i} = P_{Demand} \quad \lambda = \frac{P_{Demand} + \sum_{i=1}^{n_{gen}} \frac{\beta_i}{2\gamma_i}}{\sum_{i=1}^{n_{gen}} \frac{1}{2\gamma_i}}$$

Example

- Neglecting system losses and generator limits, find the optimal dispatch and the total cost in \$/hr for the three generators and the given load demand

$$C_1 = 500 + 5.3P_1 + 0.004P_1^2 \text{ [\$ / } MW hr]$$

$$C_2 = 400 + 5.5P_2 + 0.006P_2^2$$

$$C_3 = 200 + 5.8P_3 + 0.009P_3^2$$

$$P_{Demand} = 800 MW$$

Example

$$\lambda = \frac{P_{Demand} + \sum_{i=1}^{n_{gen}} \frac{\beta_i}{2\gamma_i}}{\sum_{i=1}^{n_{gen}} \frac{1}{2\gamma_i}} = \frac{800 + \frac{5.3}{0.008} + \frac{5.5}{0.012} + \frac{5.8}{0.018}}{\frac{1}{0.008} + \frac{1}{0.012} + \frac{1}{0.018}} = \$8.5 / MWhr$$

$$P_i = \frac{\lambda - \beta_i}{2\gamma_i}$$

$$P_1 = \frac{8.5 - 5.3}{2(0.004)} = 400MW$$

$$P_2 = \frac{8.5 - 5.5}{2(0.006)} = 250MW$$

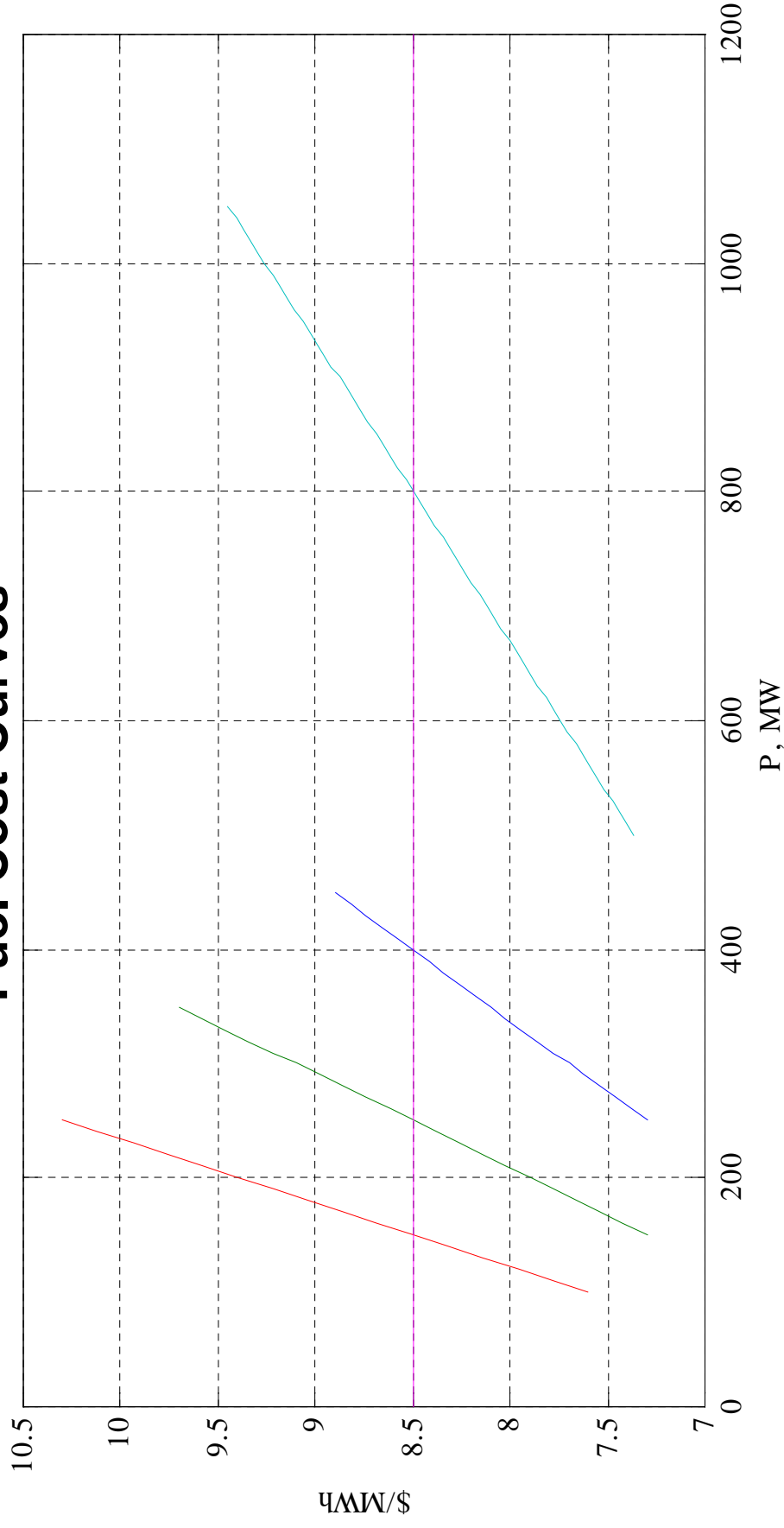
$$P_3 = \frac{8.5 - 5.8}{2(0.009)} = 150MW$$

$$P_{Demand} = 800 = 400 + 250 + 150$$

Power Systems I

Example

Fuel Cost Curves



Power Systems I

Example

- ◆ Solve again using iterative methods

$$\Delta y = \Delta f(\lambda) = P_D - f(\lambda) \quad f(\lambda) = \sum \frac{\lambda - \beta_i}{2\gamma_i}$$

$$x = \lambda \rightarrow \lambda^{[k+1]} = \lambda^{[k]} + \Delta \lambda^{[k]}$$

$$\left(\partial f / \partial \lambda \right) = \sum dP_i / d\lambda = \sum \frac{1}{2\gamma_i}$$

$$\Delta \lambda^{[k]} = \frac{\Delta f(\lambda)^{[k]}}{\left(\partial f / \partial \lambda \right)^{[k]}} = \frac{P_D - f(\lambda)}{\frac{1}{2\gamma_1} + \frac{1}{2\gamma_2} + \frac{1}{2\gamma_3}}$$

Example

$$\lambda^{[0]} = 6.0 \quad f(\lambda)^{[0]} = \sum \frac{\lambda^{[0]} - \beta_i}{2\gamma_i}$$

$$P_1^{[0]} = \frac{6.0 - 5.3}{2(0.004)} = 87.5 \quad P_2^{[0]} = \frac{6.0 - 5.5}{2(0.006)} = 41.7$$

$$P_3^{[0]} = \frac{6.0 - 5.8}{2(0.009)} = 11.1 \quad \Delta f = 800 - (87.5 + 41.7 + 11.1)$$

$$\Delta \lambda^{[0]} = \frac{\Delta f(\lambda)^{[0]}}{\left(\frac{\partial f}{\partial \lambda} \right)^{[0]}} = \frac{(659.7)}{\frac{1}{2(0.004)} + \frac{1}{2(0.006)} + \frac{1}{2(0.009)}} = 2.5$$

$$\lambda^{[1]} = 6.0 + 2.5 = 8.5$$

Power Systems I

Example

$$\lambda^{[1]} = 8.5 \quad f(\lambda)^{[1]} = \sum \frac{\lambda^{[1]} - \beta_i}{2\gamma_i}$$

$$P_1^{[1]} = \frac{8.5 - 5.3}{2(0.004)} = 400 \quad P_2^{[1]} = \frac{8.5 - 5.5}{2(0.006)} = 250$$

$$P_3^{[1]} = \frac{8.5 - 5.8}{2(0.009)} = 150 \quad \Delta f = 800 - (400 + 250 + 150)$$

$$\Delta \lambda^{[1]} = \frac{\Delta f(\lambda)^{[1]}}{\left(\frac{\partial f}{\partial \lambda} \right)^{[1]}} = \frac{(0)}{\frac{1}{2(0.004)} + \frac{1}{2(0.006)} + \frac{1}{2(0.009)}} = 0$$

$$\lambda^{[2]} = 8.5 + 0 = 8.5$$

Power Systems I