Inequality Constraints in Optimization

- **Practical problems contain inequality as well as equality constraints**
- **Minimize the cost function** $f\left(x_1, x_2, \cdots x_n\right)$
  - subject to the equality constraints

    $$g_i\left(x_1, x_2, \cdots x_n\right) = 0 \qquad i = 1, 2, \cdots, k$$
  - and the inequality constraints

    $$u_j\left(x_1, x_2, \cdots x_n\right) \le 0 \qquad j = 1, 2, \cdots, m$$
- **The Lagrange multiplier is extended to include the inequality constraints by introducing the m-dimensional vector μ of undetermined quantities**

Kuhn-Tucker Method

- **The unconstrained cost function becomes**

$$L = f + \sum_i \lambda_i g_i + \sum_j \mu_j u_j$$

- **The resulting necessary conditions for contrained local minima of L are the following**

$$\partial L / \partial x_l = 0 \quad l = 1, \cdots, n \qquad \partial L / \partial \lambda_i = g_i = 0 \quad i = 1, \cdots, k$$

$$\partial L / \partial \mu_j = u_j \le 0 \quad j = 1, \cdots, m \qquad \mu_j u_j = 0 \;\&\; \mu_j > 0 \quad j = 1, \cdots, m$$

Example

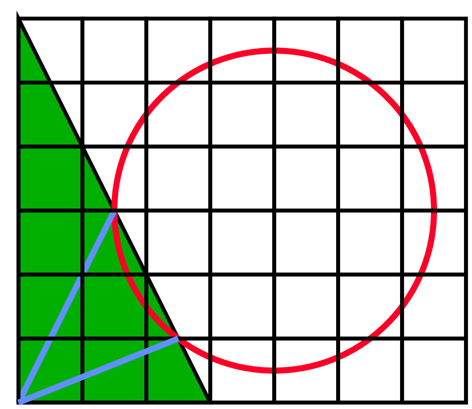
- **Use the Kuhn-Tucker method to determine the minimum distance from the origin of the x-y plane to a circle described by**

$$(x-8)^2+(y-6)^2=25 \quad or$$

$$g(x,y)=(x-8)^2+(y-6)^2-25$$

constrained by

$$u(x,y)=2x+y\geq 12$$



- The minimum distance is obtained by minimizing the distance squared

$$f(x,y)=x^2+y^2$$

Example

$$f(x,y) = x^2 + y^2 \qquad g(x,y) = (x-8)^2 + (y-6)^2 - 25 = 0$$

$$u(x,y) = 2x + y \geq 12$$

The cost function

$$L = f + \lambda \cdot g + \mu \cdot u$$

$$= x^2 + y^2 + \lambda\left[(x-8)^2 + (y-6)^2 - 25\right] + \mu[2x + y - 12]$$

The resulting necessary conditions for constrained local minima of L

$$\frac{\partial L}{\partial x} = 2x + 2\lambda(x-8) + 2\mu = 0 \qquad \frac{\partial L}{\partial y} = 2y + 2\lambda(y-6) + \mu = 0$$

$$\frac{\partial L}{\partial \lambda} = (x-8)^2 + (y-6)^2 - 25 = 0 \qquad \frac{\partial L}{\partial \mu} = 2x + y - 12 = 0$$

Example

- eliminating λ from the first two equations

$$\frac{16\lambda}{2x} = \frac{12\lambda}{2y} \rightarrow y = \frac{3}{4}x$$

- substituting for y in the third equation yields

$$(x-8)^2 + \left(\frac{3}{4}x - 6\right)^2 - 25 = 0$$

$$\frac{25}{16}x^2 - 25x + 75 = 0 \rightarrow x = 4 \quad \& \quad x = 12$$

$$extrema: \quad (4,3),\ \lambda = 1 \quad and \quad (12,9),\ \lambda = -3$$

$$\min \rightarrow \begin{matrix} x = 4 \\ y = 3 \end{matrix}$$

Economic Dispatch with Generator Limits

- **The power output of any generator should not exceed its rating nor be below the value for stable boiler operation**
  - Generators have a minimum and maximum real power output limits
- **The problem is to find the real power generation for each plant such that cost are minimized, subject to:**
  - Meeting load demand - equality constraints
  - Constrained by the generator limits - inequality constraints
- **The Kuhn-Tucker conditions**

$$dC_i / dP_i = \lambda \quad \leftarrow \quad P_{i(\min)} < P_i < P_{i(\max)}$$

$$dC_i / dP_i \le \lambda \quad \leftarrow \quad P_i = P_{i(\max)}$$

$$dC_i / dP_i \ge \lambda \quad \leftarrow \quad P_i = P_{i(\min)}$$

Example

- **Neglecting system losses, find the optimal dispatch and the total cost in $/hr for the three generators and the given load demand and generation limits**

$$C_1 = 500 + 5.3P_1 + 0.004P_1^2 \quad [\$/\mathrm{MWhr}]$$

$$C_2 = 400 + 5.5P_2 + 0.006P_2^2$$

$$C_3 = 200 + 5.8P_3 + 0.009P_3^2$$

$$200 \le P_1 \le 450$$

$$150 \le P_2 \le 350$$

$$100 \le P_3 \le 225$$

$$P_{Demand} = 975\ \mathrm{MW}$$

Example

$$\lambda = \frac{P_{Demand} + \sum_{i=1}^{n_{gen}} \frac{\beta_i}{2\gamma_i}}{\sum_{i=1}^{n_{gen}} \frac{1}{2\gamma_i}} = \frac{800 + \frac{5.3}{0.008} + \frac{5.5}{0.012} + \frac{5.8}{0.018}}{\frac{1}{0.008} + \frac{1}{0.012} + \frac{1}{0.018}} = \$8.5 / \text{MWhr}$$

$$P_i = \frac{\lambda - \beta_i}{2\gamma_i} \quad \Rightarrow \quad P_1 = \frac{8.5 - 5.3}{2(0.004)} = 400 \text{ MW}$$

$$P_2 = \frac{8.5 - 5.5_i}{2(0.006)} = 250 \text{ MW}$$

$$P_3 = \frac{8.5 - 5.8}{2(0.009)} = 150 \text{ MW}$$

$$P_{Demand} = 800 \text{ MW} = 400 + 250 + 150 \text{ MW}$$

Example

- **Solve using iterative methods**
  - initial guess $\lambda^{[1]} = 6.0$

$$P_1^{[1]} = \frac{6.0 - 5.3}{2(0.004)} = 87.5 \qquad P_2^{[1]} = \frac{6.0 - 5.5}{2(0.006)} = 41.7$$

$$P_3^{[1]} = \frac{6.0 - 5.8}{2(0.009)} = 11.1$$

$$\Delta P^{[1]} = 975 - (87.5 + 41.7 + 11.1) = 834.7$$

$$\Delta\lambda^{[1]} = \frac{\Delta P^{[1]}}{(\partial P/\partial\lambda)^{[1]}} = \frac{(834.7)}{\frac{1}{2(0.004)} + \frac{1}{2(0.006)} + \frac{1}{2(0.009)}} = 3.1632$$

$$\lambda^{[2]} = 6.0 + 3.1632 = 9.1632$$

Example

$$\lambda^{[2]} = 9.1632 \qquad P_1^{[2]} = \frac{9.16 - 5.3}{2(0.004)} = 483$$

$$P_2^{[2]} = \frac{9.16 - 5.5}{2(0.006)} = 305 \qquad P_3^{[2]} = \frac{9.16 - 5.8}{2(0.009)} = 187$$

$$\Delta P^{[2]} = 975 - (483 + 305 + 187) = 0$$

$$P_1^{[2]} = 483 > 450_{\max} \quad \rightarrow \quad P_1^{[FIXED]} = 450_{\max}$$

$$\therefore \quad \Delta P^{[2]} = 975 - (450 + 305 + 187) = 33$$

$$\Delta\lambda^{[2]} = \frac{\Delta P^{[2]}}{(\partial P / \partial \lambda)^{[2]}} = \frac{(33)}{\frac{1}{2(0.006)} + \frac{1}{2(0.009)}} = 0.2368$$

$$\lambda^{[3]} = 9.1632 + 0.2368 = 9.4$$

Example

$$\lambda^{[3]} = 9.4 \qquad P_1^{[FIXED]} = 450$$

$$P_2^{[3]} = \frac{9.4 - 5.5}{2(0.006)} = 325 \qquad P_3^{[3]} = \frac{9.4 - 5.8}{2(0.009)} = 200$$

$$\Delta P^{[3]} = 975 - (450 + 325 + 200) = 0$$

$$150 \le \left(P_2^{[3]} = 325\right) \le 350 \qquad 100 \le \left(P_3^{[3]} = 200\right) \le 225$$

$$\begin{aligned} C_{total} &= 500 + 5.3(450) + 0.004(450)^2 \\ &+ 400 + 5.5(325) + 0.006(325)^2 \\ &+ 200 + 5.8(200) + 0.009(200)^2 = 8{,}236.25 \ \$/\mathrm{hr} \end{aligned}$$

Economic Dispatch including Losses

- **For large interconnected system where power is transmitted over long distances with low load density areas**
  - transmission line losses are a major factor
  - losses affect the optimum dispatch of generation
- **One common practice for including the effect of transmission losses is to express the total transmission loss as a quadratic function of the generator power outputs**
  - simplest form: $$P_L = \sum_{i=1}^{n_{gen}} \sum_{j=1}^{n_{gen}} P_i B_{ij} P_j$$
  - Kron's loss formula: $$P_L = \sum_{i=1}^{n_{gen}} \sum_{j=1}^{n_{gen}} P_i B_{ij} P_j + \sum_{j=1}^{n_{gen}} B_{0j} P_j + B_{00}$$

Economic Dispatch including Losses

- **B_{ij} are called the loss coefficients**
  - they are assumed to be constant
  - reasonable accuracy is expected when actual operating conditions are close to the base case conditions used to compute the coefficients
- **The economic dispatch problem is to minimize the overall generation cost, C, which is a function of plant output**
- **Constraints:**
  - the generation equals the total load demand plus transmission losses
  - each plant output is within the upper and lower generation limits - inequality constraints

Economic Dispatch including Losses

$$f: \quad C_{total} = \sum_{i=1}^{n_{gen}} C_i = \sum_{i=1}^{n_{gen}} \alpha_i + \beta_i P_i + \gamma_i P_i^2$$

$$g: \quad \sum_{i=1}^{n_{gen}} P_i = P_{demand} + P_{losses}$$

$$u: \quad P_{i(\min)} \le P_i \le P_{i(\max)} \quad i = 1, \cdots, n_{gen}$$

The resulting optimization equation

$$L = C_{total} + \lambda\left(P_{demand} + P_{losses} - \sum_{i=1}^{n_{gen}} P_i \right) + \sum_{i=1}^{n_{gen}} \mu_{i(\max)} \left(P_{i(\max)} - P_i \right)$$

$$+ \sum_{i=1}^{n_{gen}} \mu_{i(\min)} \left(P_i - P_{i(\min)} \right)$$

$$P_i < P_{i(\max)}: \quad \mu_{i(\max)} = 0 \qquad\qquad P_i > P_{i(\min)}: \quad \mu_{i(\min)} = 0$$

Economic Dispatch including Losses

- **The minimum of the unconstrained function is found when:**

$$\frac{\partial L}{\partial P_i} = 0$$

$$\frac{\partial L}{\partial \lambda} = 0$$

$$\frac{\partial L}{\partial \mu_{i(\max)}} = P_i - P_{i(\max)} = 0$$

$$\frac{\partial L}{\partial \mu_{i(\min)}} = P_i - P_{i(\min)} = 0$$

Economic Dispatch including Losses

- **When generator limits are not violated:**

$$\frac{\partial L}{\partial P_i} = 0 = \frac{\partial C_{total}}{\partial P_i} + \lambda\left(0 + \frac{\partial P_L}{\partial P_i} - 1\right)$$

$$\frac{\partial C_{total}}{\partial P_i} = \frac{\partial}{\partial P_i}\left(C_1 + C_2 + \cdots + C_{n_{gen}}\right) = \frac{dC_i}{dP_i}$$

$$\therefore \quad \lambda = \frac{dC_i}{dP_i} + \lambda\frac{\partial P_L}{\partial P_i} = \left(\frac{1}{1 - \partial P_L / \partial P_i}\right)\frac{dC_i}{dP_i} = L_i \frac{dC_i}{dP_i}$$

$$\frac{\partial L}{\partial \lambda} = 0 = P_D + P_L - \sum_{i=1}^{n_{gen}} P_i \qquad \therefore \sum_{i=1}^{n_{gen}} P_i = P_D + P_L$$

The Penalty Factor

- **The incremental transmission loss equation becomes the penalty factor**

$$L_i = \frac{1}{1 - \frac{\partial P_L}{\partial P_i}}$$

  - The effect of transmission losses introduces a penalty factor that depends on the location of the plant
  - The minimum cost is obtained when the incremental cost of each plant multiplied by its penalty factor is the same for all plants

The Penalty Factor

$$P_L = \sum_{i=1}^{n_{gen}} \sum_{j=1}^{n_{gen}} P_i B_{ij} P_j + \sum_{j=1}^{n_{gen}} B_{0j} P_j + B_{00}$$

$$\frac{\partial P_L}{\partial P_i} = 2 \sum_{j=1}^{n_{gen}} B_{ij} P_j + B_{0i} \qquad \frac{dC_i}{dP_i} = \beta_i + 2\gamma_i P$$

$$\lambda = \frac{dC_i}{dP_i} + \lambda \frac{\partial P_L}{\partial P_i} = \beta_i + 2\gamma_i P_i + 2\lambda \sum_{j=1}^{n_{gen}} B_{ij} P_j + \lambda B_{0i}$$

Rearrange the equation

$$\left(\frac{\gamma_i}{\lambda} + B_{ii} \right) P_i + \sum_{\substack{j=1 \\ j \neq i}}^{n_{gen}} B_{ij} P_j = \tfrac{1}{2} \left(1 - B_{0i} - \frac{\beta_i}{\lambda} \right)$$

The Penalty Factor

- **Extending the equation to all plants results in the following linear equations (in matrix form)**

$$\begin{bmatrix} \frac{\gamma_1}{\lambda} + B_{11} & B_{12} & \cdots & B_{1n} \\ B_{21} & \frac{\gamma_2}{\lambda} + B_{22} & \cdots & B_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ B_{n1} & B_{n2} & \cdots & \frac{\gamma_n}{\lambda} + B_{nn} \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_n \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 - B_{01} - \frac{\beta_1}{\lambda} \\ 1 - B_{02} - \frac{\beta_2}{\lambda} \\ \\ 1 - B_{0n} - \frac{\beta_n}{\lambda} \end{bmatrix}$$

or

$$\mathbf{E}\,\mathbf{P} = \mathbf{D}$$

Economic Dispatch including Losses

- **To find the optimal dispatch for an estimated value of $\lambda^{[1]}$**
  - The simultaneous linear equation is solved, $\mathbf{E\,P} = \mathbf{D}$
  - Then the iterative process is continued using the gradient method
  - If an approximate loss formula is used

$$P_L = \sum_{i=1}^{n_{gen}} B_{ii}\, P_i^2$$

the solution of the simultaneous equations reduces to

$$P_i^{[k]} = \frac{\lambda^{[k]} - \beta_i}{2\left(\gamma_i + \lambda^{[k]} B_{ii}\right)^2} \qquad \sum_{i=1}^{n_{gen}} \left(\frac{\partial P_i}{\partial \lambda}\right) = \sum_{i=1}^{n_{gen}} \left[\frac{\gamma_i + B_{ii}\beta_i}{2\left(\gamma_i + \lambda^{[k]} B_{ii}\right)^2}\right]$$

Example

- **Find the optimal dispatch and the total cost in $/hr**
  - fuel costs and plant output limits

$$C_1 = 200 + 7.0P_1 + 0.008P_1^2 \text{ [\$/hr]} \qquad 10 \le P_1 \le 85 \text{ MW}$$
$$C_2 = 180 + 6.3P_2 + 0.009P_2^2 \qquad 10 \le P_2 \le 80$$
$$C_3 = 140 + 6.8P_3 + 0.007P_3^2 \qquad 10 \le P_3 \le 70$$

  - real power loss and total load demand

$$P_{loss} = 0.000218\, P_1^2 + 0.000228\, P_2^2 + 0.000179\, P_3^2$$
$$P_{Demand} = 150 \text{ MW}$$

Example

First iteration with initial guess at 8.0

$$\lambda^{[0]} = 8.0 \qquad P_i^{[k]} = \frac{\lambda^{[k]} - \beta_i}{2\left(\gamma_i + \lambda^{[k]} B_{ii}\right)^2}$$

$$P_1^{[1]} = \frac{8.0 - 7.0}{2(0.008 + 8.0 \cdot 0.000218)^2} = 51.3$$

$$P_2^{[1]} = \frac{8.0 - 6.3}{2(0.009 + 8.0 \cdot 0.000228)^2} = 78.5$$

$$P_3^{[1]} = \frac{8.0 - 6.8}{2(0.007 + 8.0 \cdot 0.000179)^2} = 71.2$$

$$P_{loss} = 0.000218(51.3)^2 + 0.000228(78.5)^2 + 0.000179(71.2)^2$$

$$P_{loss} = 2.886$$

Example

$$\Delta P^{[k]} = P_D + P_L^{[k]} - \left(P_1^{[k]} + P_2^{[k]} + P_3^{[k]}\right)$$ **Newton-Raphson**

$$\Delta P^{[1]} = 150 + 2.9 - (51.3 + 78.5 + 71.2) = -48.1$$

$$J^{[k]} = \sum_{i=1}^{3} \left(\frac{\partial P_i}{\partial \lambda}\right)^{[k]} = \sum_{i=1}^{3} \frac{\gamma_i + B_{ii}\beta_i}{2\left(\gamma_i + \lambda^{[k]} B_{ii}\right)^2}$$ **the Jacobian**

$$J^{[1]} = \frac{0.008 + 0.000218 \times 7.0}{2(0.008 + 8.0 \times 0.000218)^2} + \frac{0.009 + 0.000228 \times 6.3}{2(0.009 + 8.0 \times 0.000228)^2} +$$

$$\frac{0.007 + 0.000179 \times 6.8}{2(0.007 + 8.0 \times 0.000179)^2} = 152.5$$

$$\Delta \lambda^{[1]} = \frac{-48.1}{152.5} = -0.316$$ **the improvement**

$$\lambda^{[2]} = \lambda^{[1]} + \Delta \lambda^{[1]} = 8.0 - 0.316 = 7.684$$

Example

Second iteration

$$\lambda^{[2]} = 7.684 \qquad P_i^{[k]} = \frac{\lambda^{[k]} - \beta_i}{2\left(\gamma_i + \lambda^{[k]} B_{ii}\right)^2}$$

$$P_1^{[2]} = \frac{7.684 - 7.0}{2\left(0.008 + 7.684 \cdot 0.000218\right)^2} = 35.4$$

$$P_2^{[2]} = \frac{7.684 - 6.3}{2\left(0.009 + 7.684 \cdot 0.000228\right)^2} = 64.4$$

$$P_3^{[2]} = \frac{7.684 - 6.8}{2\left(0.007 + 7.684 \cdot 0.000179\right)^2} = 52.8$$

$$P_{loss} = 0.000218\left(35.4\right)^2 + 0.000228\left(64.4\right)^2 + 0.000179\left(52.8\right)^2$$

$$P_{loss} = 1.717$$

Example

$$\Delta P^{[k]} = P_D + P_L^{[k]} - \left(P_1^{[k]} + P_2^{[k]} + P_3^{[k]}\right)$$ **Newton-Raphson**

$$\Delta P^{[2]} = 150 + 1.7 - (35.4 + 64.4 + 52.8) = -0.84$$

$$J^{[k]} = \sum_{i=1}^{3} \left(\frac{\partial P_i}{\partial \lambda}\right)^{[k]} = \sum_{i=1}^{3} \frac{\gamma_i + B_{ii}\beta_i}{2\left(\gamma_i + \lambda^{[k]} B_{ii}\right)^2}$$ **the Jacobian**

$$J^{[2]} = \frac{0.008 + 0.000218 \times 7.0}{2(0.008 + 7.68 \times 0.000218)^2} + \frac{0.009 + 0.000228 \times 6.3}{2(0.009 + 7.68 \times 0.000228)^2} + \frac{0.007 + 0.000179 \times 6.8}{2(0.007 + 7.68 \times 0.000179)^2} = 154.6$$

$$\Delta \lambda^{[2]} = \frac{-0.84}{154.6} = -0.00543$$ **the improvement**

$$\lambda^{[3]} = \lambda^{[2]} + \Delta \lambda^{[2]} = 7.684 - 0.0.00543 = 7.679$$

Example

Third iteration

$$\lambda^{[3]} = 7.679 \qquad P_i^{[k]} = \frac{\lambda^{[k]} - \beta_i}{2\left(\gamma_i + \lambda^{[k]} B_{ii}\right)^2}$$

$$P_1^{[3]} = \frac{7.679 - 7.0}{2\left(0.008 + 7.679 \cdot 0.000218\right)^2} = 35.09$$

$$P_2^{[3]} = \frac{7.679 - 6.3}{2\left(0.009 + 7.679 \cdot 0.000228\right)^2} = 64.14$$

$$P_3^{[3]} = \frac{7.679 - 6.8}{2\left(0.007 + 7.679 \cdot 0.000179\right)^2} = 52.48$$

$$P_{loss} = 0.000218\left(35.09\right)^2 + 0.000228\left(64.14\right)^2 + 0.000179\left(52.48\right)^2$$

$$P_{loss} = 1.699$$

Example

$$\Delta P^{[k]} = P_D + P_L^{[k]} - \left(P_1^{[k]} + P_2^{[k]} + P_3^{[k]}\right)$$ **Newton-Raphson**

$$\Delta P^{[3]} = 150 + 1.697 - (35.09 + 64.14 + 52.48) = -0.0174$$

$$J^{[k]} = \sum_{i=1}^{3} \left(\frac{\partial P_i}{\partial \lambda}\right)^{[k]} = \sum_{i=1}^{3} \frac{\gamma_i + B_{ii}\beta_i}{2\left(\gamma_i + \lambda^{[k]} B_{ii}\right)^2}$$ **the Jacobian**

$$J^{[3]} = \frac{0.008 + 0.000218 \times 7.0}{2(0.008 + 7.679 \times 0.000218)^2} + \frac{0.009 + 0.000228 \times 6.3}{2(0.009 + 7.679 \times 0.000228)^2} + \frac{0.007 + 0.000179 \times 6.8}{2(0.007 + 7.679 \times 0.000179)^2} = 154.6$$

$$\Delta \lambda^{[3]} = \frac{-0.0174}{154.6} = -0.000113$$ **the improvement**

$$\lambda^{[4]} = \lambda^{[3]} + \Delta \lambda^{[3]} = 7.684 - 0.0.000113 = 7.6789$$

Example

Forth iteration

$$\lambda^{[4]} = 7.6789 \qquad P_i^{[k]} = \frac{\lambda^{[k]} - \beta_i}{2\left(\gamma_i + \lambda^{[k]} B_{ii}\right)^2}$$

$$P_1^{[3]} = \frac{7.6789 - 7.0}{2\left(0.008 + 7.6789 \cdot 0.000218\right)^2} = 35.09$$

$$P_2^{[3]} = \frac{7.6789 - 6.3}{2\left(0.009 + 7.6789 \cdot 0.000228\right)^2} = 64.13$$

$$P_3^{[3]} = \frac{7.6789 - 6.8}{2\left(0.007 + 7.6789 \cdot 0.000179\right)^2} = 52.47$$

$$P_{loss} = 0.000218\left(35.09\right)^2 + 0.000228\left(64.13\right)^2 + 0.000179\left(52.47\right)^2$$

$$P_{loss} = 1.699$$

Example

$$\Delta P^{[k]} = P_D + P_L^{[k]} - \left(P_1^{[k]} + P_2^{[k]} + P_3^{[k]}\right)$$

$$\Delta P^{[2]} = 150 + 1.699 - (35.09 + 64.13 + 52.47) = 0.0$$

$$\begin{aligned} C_{total} &= 200 + 7.0(35.09) + 0.008(35.09)^2 \\ &\quad + 180 + 6.3(64.13) + 0.009(64.13)^2 \\ &\quad + 140 + 6.8(52.47) + 0.007(52.47)^2 \\ &= 1592.65 \quad \$/\mathrm{hr} \end{aligned}$$