The Bus Impedance Matrix

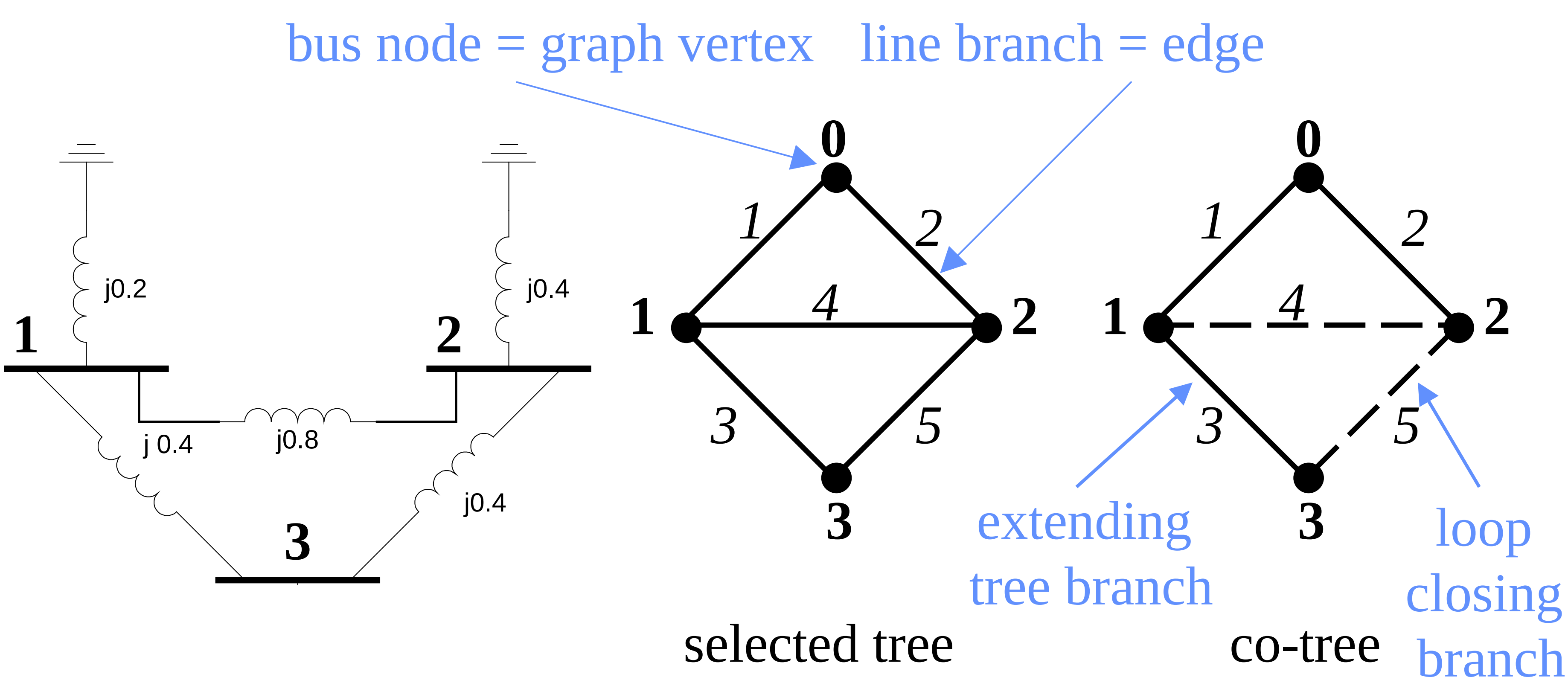
- **Definition**

$$\mathbf{Z}_{bus} = \mathbf{Y}_{bus}^{-1}$$

- **Direct formation of the matrix**
  - inversion of the bus admittance matrix is a $\boldsymbol{n}^3$ effort
  - for small and medium size networks, direct building of the matrix is less effort
  - for large size networks, sparse matrix programming with gaussian elimination technique is preferred

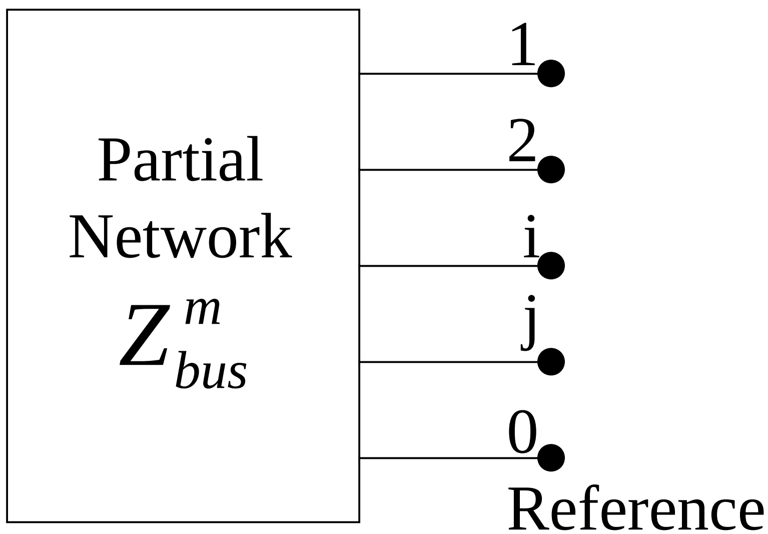
Forming the Bus Impedance Matrix

- **Graph theory techniques helps explain the building process**



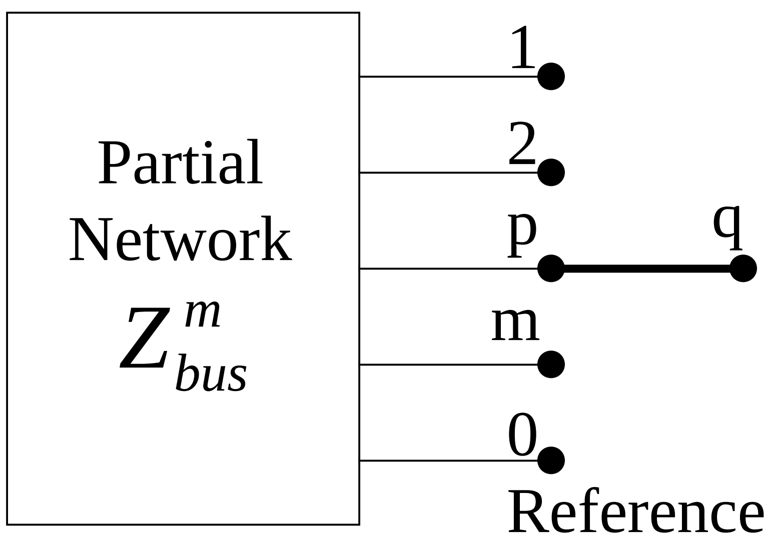
Forming the Bus Impedance Matrix

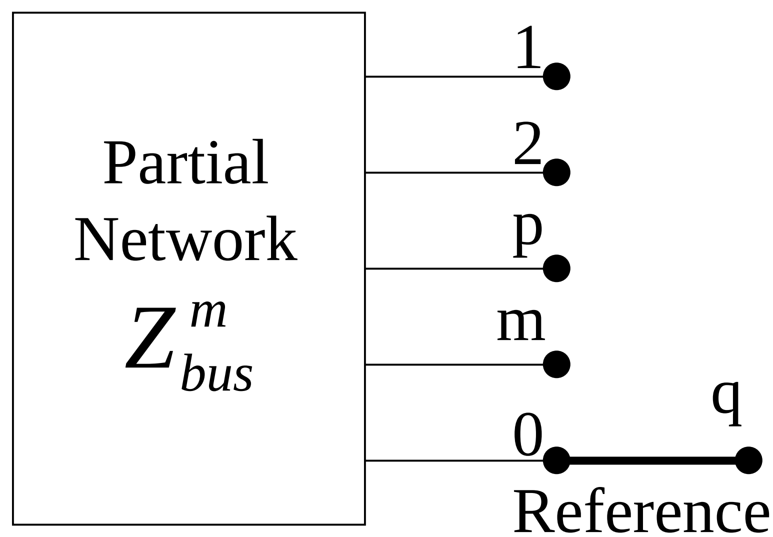
- **Basic construction of the network and the matrix**



$$V_{bus} = Z_{bus} I_{bus}$$

Adding a Line





$$V_q = V_p + z_{qp} I_q$$

$$V_q = 0 + z_{q0} I_q$$

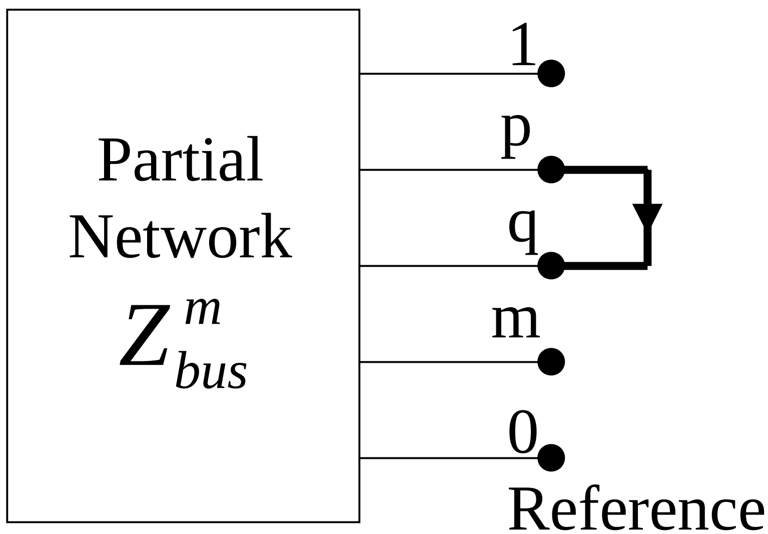
Adding a Line to an Existing Line

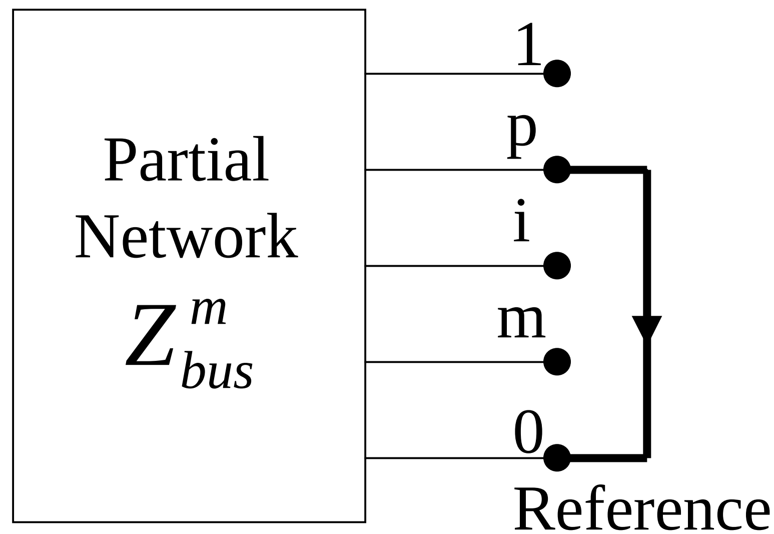
$$
\begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_p \\ \vdots \\ V_m \\ V_q \end{bmatrix}
=
\left[\begin{array}{cccccc|c}
Z_{11} & Z_{21} & \cdots & Z_{1p} & \cdots & Z_{1m} & = Z_{1p} \\
Z_{21} & Z_{22} & \cdots & Z_{2p} & \cdots & Z_{2m} & = Z_{2p} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
Z_{p1} & Z_{p2} & \cdots & Z_{pp} & \cdots & Z_{pm} & = Z_{pp} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
Z_{m1} & Z_{m2} & \cdots & Z_{mp} & \cdots & Z_{mm} & = Z_{mp} \\
\hline
= Z_{p1} & = Z_{p2} & \cdots & = Z_{pp} & \cdots & = Z_{pm} & = Z_{pp} + z_{pq}
\end{array}\right]
\begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_p \\ \vdots \\ I_m \\ I_q \end{bmatrix}
$$

Adding a Line from Reference

$$
\begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_p \\ \vdots \\ V_m \\ \hline V_q \end{bmatrix}
=
\left[\begin{array}{cccccc|c}
Z_{11} & Z_{21} & \cdots & Z_{1p} & \cdots & Z_{1m} & =0 \\
Z_{21} & Z_{22} & \cdots & Z_{2p} & \cdots & Z_{2m} & =0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
Z_{p1} & Z_{p2} & \cdots & Z_{pp} & \cdots & Z_{pm} & =0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
Z_{m1} & Z_{m2} & \cdots & Z_{mp} & \cdots & Z_{mm} & =0 \\
\hline
=0 & =0 & \cdots & =0 & \cdots & =0 & =z_{0q}
\end{array}\right]
\begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_p \\ \vdots \\ I_m \\ \hline I_q \end{bmatrix}
$$

Closing a Loop





$$z_{pq} I_l = V_p - V_q \rightarrow$$

$$z_{pq} I_l + V_q - V_p = 0$$

$$z_{p0} I_l = V_p - 0 \rightarrow$$

$$z_{p0} I_l - V_p = 0$$

Kron Reduction

Eliminating a node from the system

$$\begin{bmatrix} V_{bus\,[1\times n]} \\ 0 \end{bmatrix} = \begin{bmatrix} Z^{old}_{bus\,[n\times n]} & \Delta Z_{[1\times n]} \\ \Delta Z^{T}{}_{[n\times 1]} & Z_{ll\,[1\times 1]} \end{bmatrix} \begin{bmatrix} I_{bus\,[1\times n]} \\ I_l \end{bmatrix}$$

$$V_{bus\,[1\times n]} = Z^{old}_{bus\,[n\times n]} I_{bus\,[1\times n]} + \Delta Z_{[1\times n]} I_l$$

$$0 = \Delta Z^{T}{}_{[n\times 1]} I_{bus\,[1\times n]} + Z_{ll\,[1\times 1]} I_l \quad \rightarrow \quad I_l = -\frac{\Delta Z^{T}{}_{[n\times 1]}}{Z_{ll}} I_{bus\,[1\times n]}$$

$$V_{bus\,[1\times n]} = Z^{old}_{bus\,[n\times n]} I_{bus\,[1\times n]} - \frac{\Delta Z_{[1\times n]} \Delta Z^{T}{}_{[n\times 1]}}{Z_{ll}} I_{bus\,[1\times n]} = \left[Z^{old}_{bus} - \frac{\Delta Z \Delta Z^{T}}{Z_{ll}} \right] I_{bus}$$

Adding a Line between two Lines

$$
\begin{bmatrix} V_1 \\ \vdots \\ V_p \\ V_q \\ \vdots \\ V_m \\ 0 \end{bmatrix}
=
\left[\begin{array}{cccccc|c}
Z_{11} & \cdots & Z_{1p} & Z_{1q} & \cdots & Z_{1m} & Z_{1q}-Z_{1p} \\
\vdots & \ddots & \vdots & \vdots & \cdots & \vdots & \vdots \\
Z_{p1} & \cdots & Z_{pp} & Z_{pq} & \ddots & Z_{pm} & Z_{pq}-Z_{pp} \\
Z_{q1} & \cdots & Z_{qp} & Z_{qq} & \cdots & Z_{qm} & Z_{qq}-Z_{qp} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
Z_{m1} & \cdots & Z_{mp} & Z_{mq} & \cdots & Z_{mm} & Z_{mq}-Z_{mp} \\
\hline
Z_{q1}-Z_{p1} & \cdots & Z_{qp}-Z_{pp} & Z_{qq}-Z_{pq} & \cdots & Z_{qm}-Z_{pm} & Z_{ll}
\end{array}\right]
\begin{bmatrix} I_1 \\ \vdots \\ I_p \\ I_q \\ \vdots \\ I_m \\ I_l \end{bmatrix}
$$

$$Z_{ll} = z_{pq} + Z_{pp} + Z_{qq} - 2\,Z_{pq}$$

Then execute Kron reduction on Z_{ll}

Adding a Line from a Line to Reference

$$
\begin{bmatrix} V_1 \\ \vdots \\ V_p \\ V_i \\ \vdots \\ V_m \\ 0 \end{bmatrix}
=
\left[\begin{array}{ccccccc|c}
Z_{11} & \cdots & Z_{1p} & Z_{1i} & \cdots & Z_{1m} & & -Z_{1p} \\
\vdots & \ddots & \vdots & \vdots & \cdots & \vdots & & \vdots \\
Z_{p1} & \cdots & Z_{pp} & Z_{pi} & \ddots & Z_{pm} & & -Z_{pp} \\
Z_{i1} & \cdots & Z_{ip} & Z_{ii} & \cdots & Z_{im} & & -Z_{ip} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & & \vdots \\
Z_{m1} & \cdots & Z_{mp} & Z_{mi} & \cdots & Z_{mm} & & -Z_{mp} \\
\hline
-Z_{p1} & \cdots & -Z_{pp} & -Z_{pi} & \cdots & -Z_{pm} & & Z_{ll}
\end{array}\right]
\begin{bmatrix} I_1 \\ \vdots \\ I_p \\ I_q \\ \vdots \\ I_m \\ I_l \end{bmatrix}
$$

$$Z_{ll} = z_{p0} + Z_{pp}$$

Then execute Kron reduction on Z_{ll}

Kron Reduction

- **Kron reduction removes an axis (row & column) from a matrix while retaining the axis's numerical influence**

$$\begin{bmatrix} V_1 \\ \vdots \\ V_p \\ V_i \\ \vdots \\ V_m \\ 0 \end{bmatrix} = \begin{bmatrix} Z_{11} & \cdots & Z_{1p} & Z_{1i} & \cdots & Z_{1m} & Z_{1l} \\ \vdots & \ddots & \vdots & \vdots & \cdots & \vdots & \vdots \\ Z_{p1} & \cdots & Z_{pp} & Z_{pi} & \ddots & Z_{pm} & Z_{pl} \\ Z_{i1} & \cdots & Z_{ip} & Z_{ii} & \cdots & Z_{im} & Z_{il} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ Z_{m1} & \cdots & Z_{mp} & Z_{mi} & \cdots & Z_{mm} & Z_{ml} \\ Z_{l1} & \cdots & Z_{lp} & Z_{li} & \cdots & Z_{lm} & Z_{ll} \end{bmatrix} \begin{bmatrix} I_1 \\ \vdots \\ I_p \\ I_q \\ \vdots \\ I_m \\ I_l \end{bmatrix}$$

$$Z_{jk}^{new} = Z_{jk}^{old} - \frac{Z_{jl} Z_{lk}}{Z_{ll}}$$

Z-Bus Building Rules

- **Rule 1: Addition of a branch to the reference**
  - start with existing network matrix $[m \times m]$
  - create a new network matrix $[(m+1) \times (m+1)]$ with
    - the new off-diagonal row and column filled with (0)
    - the diagonal element $(m+1),(m+1)$ filled with the element impedance z_{q0}

Z-Bus Building Rules

- **Rule 2: Addition of a branch to an existing bus**
  - connecting to existing bus p
  - start with existing network matrix $[m \times m]$
  - create a new network matrix $[(m+1) \times (m+1)]$ with
    - the new off-diagonal row and column filled with a copy of row p and column p
    - the diagonal element (m+1),(m+1) filled with the element impedance z_{pq} plus the diagonal impedance Z_{pp}

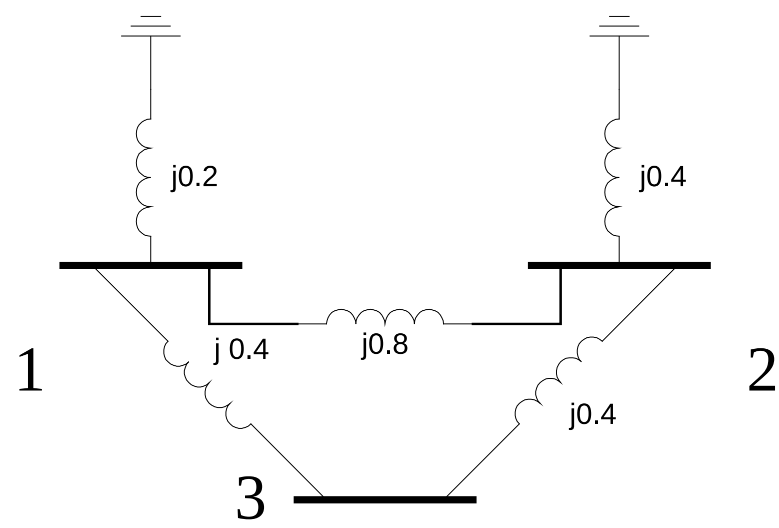
Z-Bus Building Rules

- **Rule 3: Addition of a linking branch**
  - connecting to existing buses p and q
  - start with existing network matrix $[m \times m]$
  - create a new network matrix $[(m+1) \times (m+1)]$ with
    - the new off-diagonal row and column filled with a copy of row q minus row p and column q minus column p
    - the diagonal element (m+1),(m+1) filled with $z_{pq} + Z_{pp} + Z_{qq} - 2\, Z_{pq}$
  - perform Kron reduction on the m+1 row and column

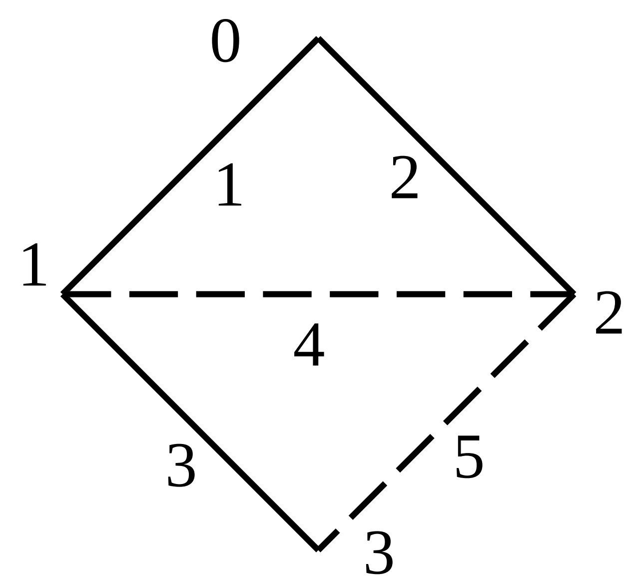
Z-Bus Building Rules

- **Rule 4: Addition of a linking branch**
  - connecting to existing bus p and reference
  - start with existing network matrix $[m \times m]$
  - create a new network matrix $[(m+1) \times (m+1)]$ with
    - the new off-diagonal row and column filled with a copy of the negative of row p and the negative of column p
    - the diagonal element (m+1),(m+1) filled with $z_{p0} + Z_{pp}$
  - perform Kron reduction on the m+1 row and column

Example



Network



Graph

Line adding order: 1-0, 2-0, 1-3, 1-2, then 2-3

Example

0. $\begin{bmatrix} \end{bmatrix}$

1. $\begin{bmatrix} j0.2 \end{bmatrix}$

2. $$\begin{bmatrix} j0.2 & 0 \\ 0 & j0.4 \end{bmatrix}$$

3. $$\begin{bmatrix} j0.2 & 0 & j0.2 \\ 0 & j0.4 & 0 \\ j0.2 & 0 & j0.6 \end{bmatrix}$$

4. $$\begin{bmatrix} j0.2 & 0 & j0.2 & j0.2 \\ 0 & j0.4 & 0 & -j0.4 \\ j0.2 & 0 & j0.6 & j0.2 \\ j0.2 & -j0.4 & j0.2 & j1.4 \end{bmatrix}$$

$$Z_{11} = j0.2 - \frac{(j0.2)(j0.2)}{j1.4} = j0.17$$

$$\begin{bmatrix} j0.171 & j0.057 & j0.171 \\ j0.057 & 0.285 & j0.057 \\ j0.171 & j0.057 & j0.571 \end{bmatrix}$$

Example

5. $$\begin{bmatrix} j0.171 & j0.057 & j0.171 & j0.114 \\ j0.057 & j0.285 & j0.057 & -j0.228 \\ j0.171 & j0.057 & j0.571 & j0.514 \\ j0.114 & -j0.228 & j0.514 & j1.14 \end{bmatrix}$$

$$\begin{bmatrix} j0.16 & j0.08 & j0.12 \\ j0.08 & j0.24 & j0.16 \\ j0.12 & j0.16 & j0.34 \end{bmatrix}$$