Solving Non-linear ODE

- **Objective**
  - Time domain solution of a system of differential equations
    - Given a function or a system of functions: $\boldsymbol{f(x)}$ or $\mathbf{F(x)}$
    - Seek a time domain solution $\boldsymbol{x(t)}$ or $\mathbf{x}(\boldsymbol{t})$ which satisfy $\boldsymbol{f(x)}$ or $\mathbf{F(x)}$
- **Integration of the differential equations**
  - Linear equations - Closed form solutions:
    - Laplace transforms
  - Non-linear equations - Frequently no closed form solutions:
    - Numerical integration
      - Taylor Series
      - Euler
      - Runga-Kutta

Solving Non-linear ODE

- **Taylor Series**
  - Consider $$\frac{dx}{dt} = f(x)$$
  - Then by expansion

$$x(t+h) = x + hx' + \frac{h}{2!}x'' + \frac{h}{3!}x''' + \frac{h}{4!}x^{iv} + \cdots$$

$$x' = -k_1 x - k_2 - c_1 t - c_2 t^2 - c_3 t^3 - \ldots$$

$$x'' = -k_1 x' - c_1 - 2c_2 t - 3c_3 t^2 - \ldots$$

$$x''' = -k_1 x'' - 2c_2 - 6c_3 t - \ldots$$

Solving Non-linear ODE

- **Euler's Method**
  - First term of the Taylor's series only is used

$$x(t+h) = x + hx' + e(t,h)$$

$$y(x) = x' + c_{01}x + c_{00} + c_{10}t + c_{20}t^2 + c_{30}t^3 + \ldots = 0$$

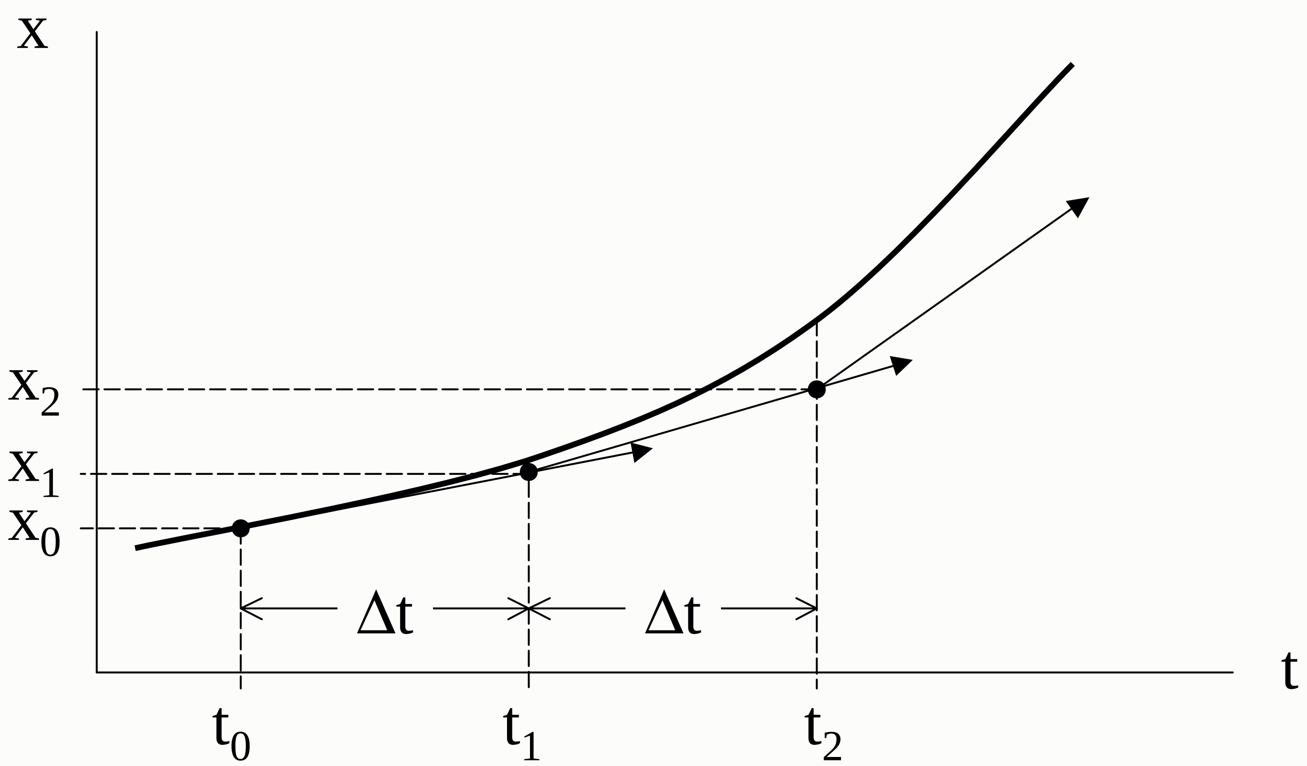
$$x' = -c_{01}x - c_{00} - c_{10}t - c_{20}t^2 - c_{30}t^3 - \ldots$$

$$t^{k+1} = t^k + h$$

$$\begin{aligned} x^{k+1} &= x^k + h\left(-c_{01}x^k - c_{00} - c_{10}t - c_{20}t^2 - c_{30}t^3 - \ldots\right) \\ &= x^k + h\left(-c_{01}x^k - c_{00} - c_{10}kh - c_{20}k^2h^2 - c_{30}k^3h^3 - \ldots\right) \end{aligned}$$

$$x(\hat{t}) = x(kh) \approx x^k$$

Euler's Method



Solving Non-linear ODE

- **Runga-Kutta**

$$x(t+h) = x + \frac{h}{6}\left(g_1 + 2g_2 + 2g_3 + g_4\right)$$

$$g_1 = x'(t,x) \qquad g_2 = x'(t, x + \tfrac{1}{2}hg_1)$$

$$g_3 = x'(t, x + \tfrac{1}{2}hg_2) \qquad g_4 = x'(t, x + hg_3)$$

$$x'(t,x) \leftarrow y(x)$$

$$t^{k+1} = t^k + h$$

$$x(\hat{t}) = x(kh) \approx x^k$$

Application

- **Power system equation**

$$\frac{H}{\pi\ f_0}\frac{d^2\delta_i}{dt^2} = P_m - P_e(\boldsymbol{\delta}) \quad \forall i \in n_{gen}$$

$$\frac{H}{\pi\ f_0} = M$$

$$\frac{d^2\delta_i}{dt^2} = \frac{P_m}{M} - \frac{P_e(\boldsymbol{\delta})}{M} = \frac{d\omega_i}{dt} \qquad \frac{d\omega_i}{dt} = x'_{1i} = \frac{P_m}{M} - \frac{P_e(\mathbf{x}_2)}{M}$$

$$\omega_i = \frac{d\delta_i}{dt} \qquad \frac{d\delta_i}{dt} = x'_{2i} = x_{1i} = \omega_i$$

Application

$$x_{1i}^{k+1} = x_{1i}^{k} + \frac{h}{6}\left(g_{1i1} + 2g_{1i2} + 2g_{1i3} + g_{1i4}\right)$$

$$g_{1i1} = x'(t,x) = \frac{P_m}{M} - \frac{P_e(\mathbf{x}_2^k)}{M}$$

$$g_{1i2} = x'(t, x + \tfrac{1}{2}hg_1) = \frac{P_m}{M} - \frac{P_e(\mathbf{x}_2^k + \tfrac{1}{2}h\mathbf{g}_{21})}{M}$$

$$g_{1i3} = x'(t, x + \tfrac{1}{2}hg_2) = \frac{P_m}{M} - \frac{P_e(\mathbf{x}_2^k + \tfrac{1}{2}h\mathbf{g}_{22})}{M}$$

$$g_{1i4} = x'(t, x + hg_3) = \frac{P_m}{M} - \frac{P_e(\mathbf{x}_2^k + h\mathbf{g}_{23})}{M}$$

Application

$$x_{2i}^{k+1} = x_{2i}^{k} + \frac{h}{6}\left(g_{2i1} + 2g_{2i2} + 2g_{2i3} + g_{2i4}\right)$$

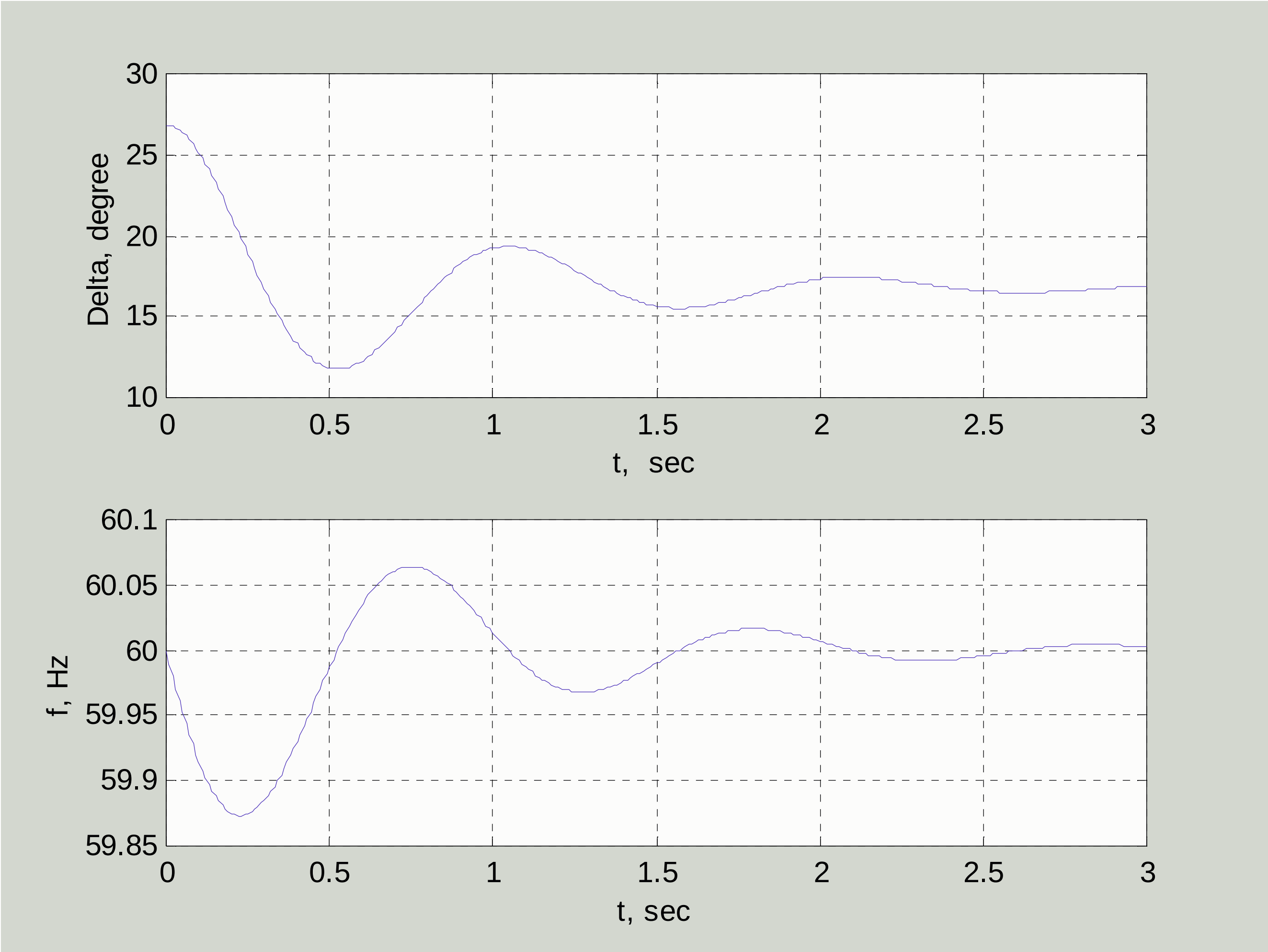
$$g_{2i1} = x'(t, x) = x_{1i}^{k}$$

$$g_{2i2} = x'(t, x + \tfrac{1}{2}hg_1) = x_{1i}^{k}$$

$$g_{2i3} = x'(t, x + \tfrac{1}{2}hg_2) = x_{1i}^{k}$$

$$g_{2i4} = x'(t, x + hg_3) = x_{1i}^{k}$$

Example



30
25
20
15
10
Delta, degree
0
0.5
1
1.5
2
2.5
3
t, sec
60.1
60.05
60
59.95
59.9
59.85
f, Hz
0
0.5
1
1.5
2
2.5
3
t, sec

Modeling Steps

- **Solve the initial load flow and obtain the initial bus voltage magnitude and phase angle**
- **Calculate the machine currents prior to the disturbance**

$$I_{mach-i} = \frac{S^*_{mach-i}}{V^*_{mach-i}}$$

- **Obtain the voltages behind the transient reactances**

$$E'_{mach-i} = V_{mach-i} + j X'_d I_{mach-i}$$

- **Convert all loads to equivalent admittances**

$$y_{i0} = \frac{S_i^*}{|V_i|^2} = \frac{P_i - jQ_i}{|V_i|^2}$$

Modeling Steps

- **Combine the generator models with the network's bus admittance matrix with converted loads**

$$\begin{bmatrix} \mathbf{0} \\ \mathbf{I}_{mach} \end{bmatrix} = \begin{bmatrix} \mathbf{Y}_{nn} & \mathbf{Y}_{n-mach} \\ \mathbf{Y}_{n-mach}^{T} & \mathbf{Y}_{mach-mach} \end{bmatrix} \begin{bmatrix} \mathbf{V}_{n} \\ \mathbf{E}'_{mach} \end{bmatrix}$$

Use kron reduction (matrix form) to remove the network buses from the matrix

$$\mathbf{Y}_{bus}^{reduced} = \mathbf{Y}_{mach-mach} - \mathbf{Y}_{n-mach}^{T} \left[\mathbf{Y}_{nn}\right]^{-1} \mathbf{Y}_{n-mach}$$

$$\mathbf{I}_{mach} = \mathbf{Y}_{bus}^{reduced} \mathbf{E}'_{mach}$$

Modeling Steps

- **Express in terms of the machines' excitation voltages, the power output**

$$S^*_{mach-i} = E'^*_{mach-i} I_{mach-i}$$

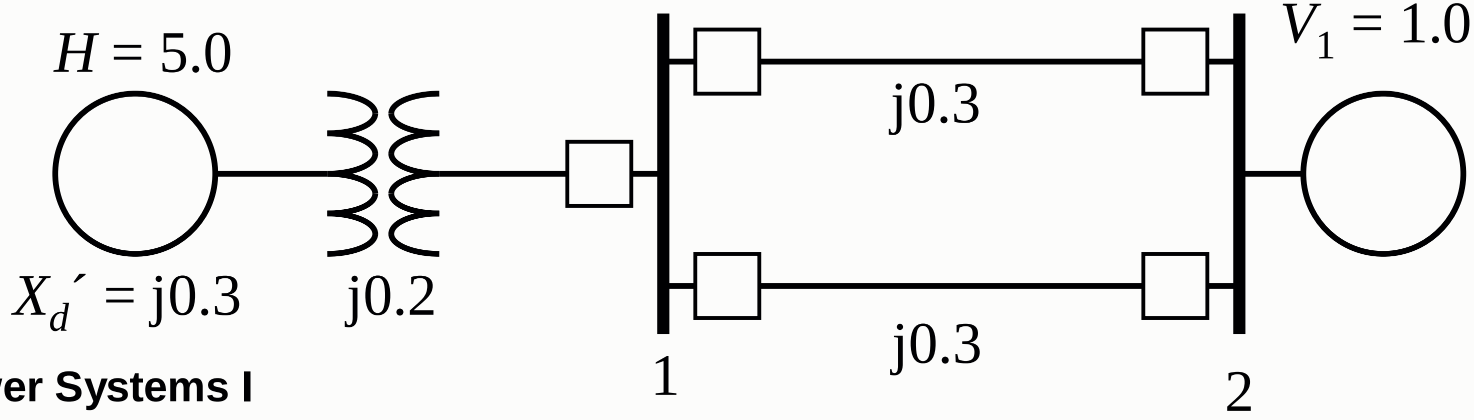
$$P_{mach-i} = \Re\left[E'^*_{mach-i} I_{mach-i}\right]$$

$$I_{mach-i} = \sum_{j=1}^{m} E'_{mach-j} Y_{ij}$$

$$P_{mach-i} = \sum_{j=1}^{m} \left|E'_{mach-i}\right|\left|E'_{mach-j}\right|\left|Y_{ij}\right| \cos\left(\theta_{ij} - \delta_i + \delta_j\right)$$

Example

- **Consider the following system**
  - a three-phase fault at the middle of one line is cleared by isolating the faulted circuit simultaneously at both ends.
  - The fault is cleared in 0.3 seconds, perform several steps of the numerical solution of the swing equation using the modified Euler method with a step size of
    $\Delta t = 0.01$ seconds.
  - graph the swing equation for clearing times of 0.3 s, 0.4 s, and 0.5 s.



Example

$H = 5$ Machine parameters

$P_m = 0.8$

$$E = V + jX_1 I = 1.0 + (j0.65)\frac{0.8 - j0.074}{1.0} = 1.17\angle 26°$$

$$P_m = P_{\max}\sin\delta = \frac{(1.17)(1.0)}{0.65}\sin\delta = 1.8\sin\delta$$ Pre-fault equation

$\delta_0 = 26.4° = 0.4606 \text{ rad}$ Initial conditions

$P_m = 0.8 \quad \Delta\omega = 0 \text{ rad/s}$

$$P_{\max}^{[fault]} = \frac{(1.17)(1.0)}{1.8}\sin\delta = 0.65\sin\delta$$

fault parameters

$$P_a = P_m - P_e^{[fault]} = 0.8 - 0.65\sin\delta$$

Example

$$\left.\frac{d\delta}{dt}\right|_{\Delta\omega_0} = 0 \text{ rad/s}$$

$$\left.\frac{d\Delta\omega}{dt}\right|_{\delta_0} = \frac{\pi \cdot 60}{5}\left(0.8 - 0.65\sin\left(0.4606 \text{ rad}\right)\right) = 19.27 \text{ rad/s}^2$$

$$\delta_1^p = 0.4606 + (0)(0.01) = 0.4606 \text{ rad}$$

$$\Delta\omega_1^p = 0 + (19.27)(0.01) = 0.1927 \text{ rad/s}$$

$$\left.\frac{d\delta}{dt}\right|_{\Delta\omega_1^p} = 0.1927 \text{ rad/s}$$

$$\left.\frac{d\Delta\omega}{dt}\right|_{\delta_1^p} = \frac{\pi \cdot 60}{5}\left(0.8 - 0.65\sin\left(0.4606 \text{ rad}\right)\right) = 19.27 \text{ rad/s}^2$$

Example

$$\delta_1^c = 0.4606 + \tfrac{1}{2}(0 + 0.1927)(0.01) = 0.4615\,\text{rad}$$

$$\Delta\omega_1^c = 0 + \tfrac{1}{2}(19.27 + 19.27)(0.01) = 0.1927\ \text{rad/s}$$

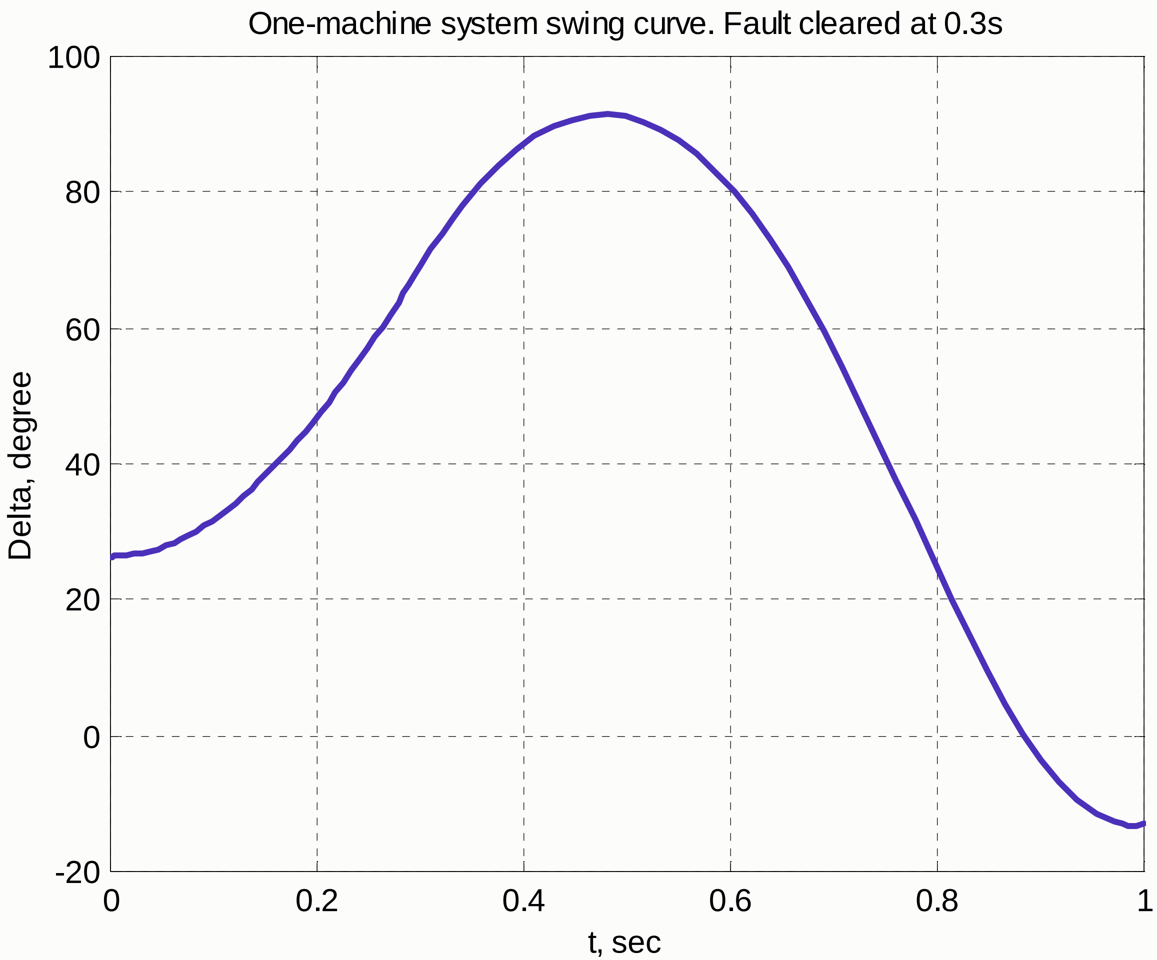
End of first step. Next step:

$$\left.\frac{d\delta}{dt}\right|_{\Delta\omega_1} = 0.1927\,\text{rad/s}$$

$$\left.\frac{d\Delta\omega}{dt}\right|_{\delta_1^p} = \frac{\pi\cdot 60}{5}\left(0.8 - 0.65\sin\left(0.4615\,\text{rad}\right)\right) = 19.25\,\text{rad/s}^2$$

Example

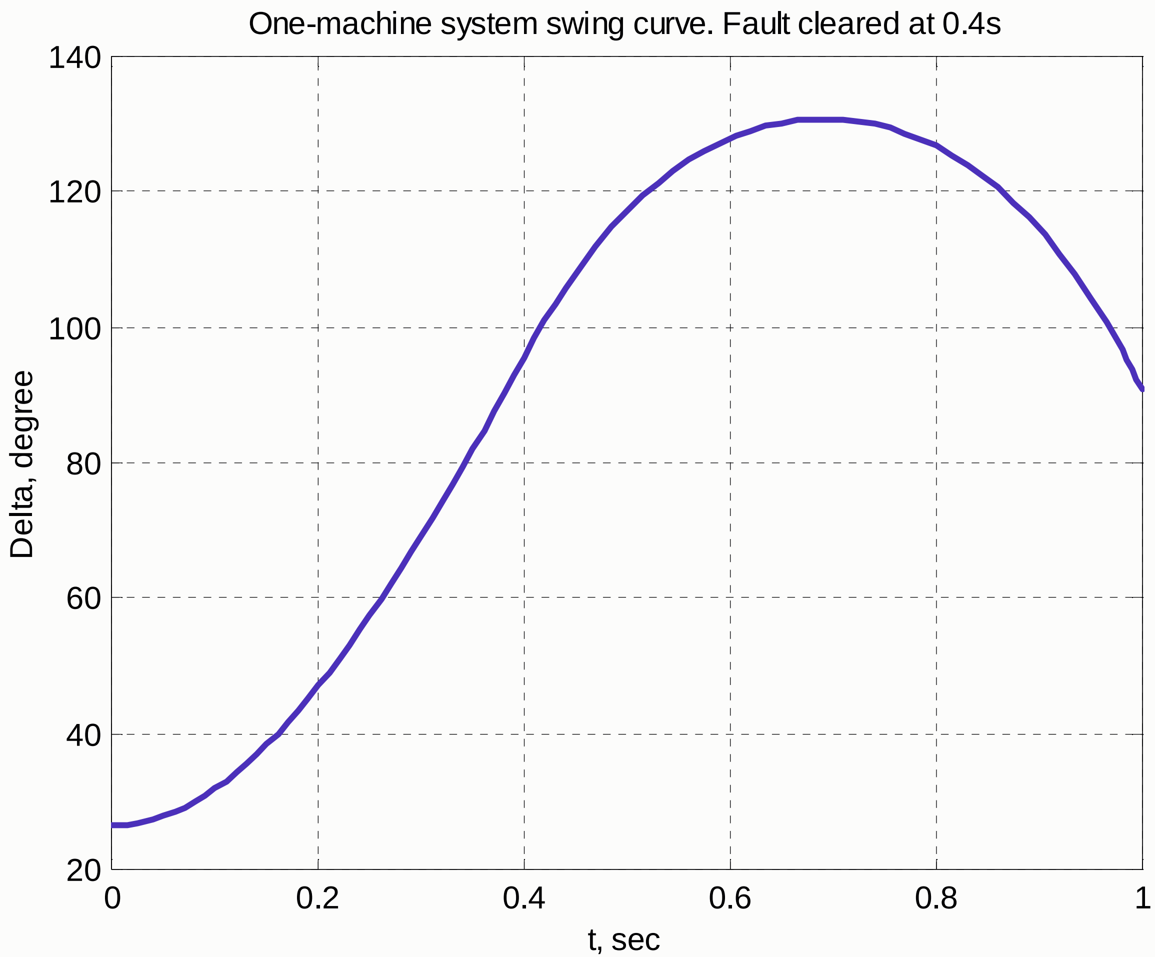
Power angle / time fault clearing in 0.3 s



Example

Power angle / time

fault clearing in 0.4 s



Example

Power angle / time fault clearing in 0.5 s