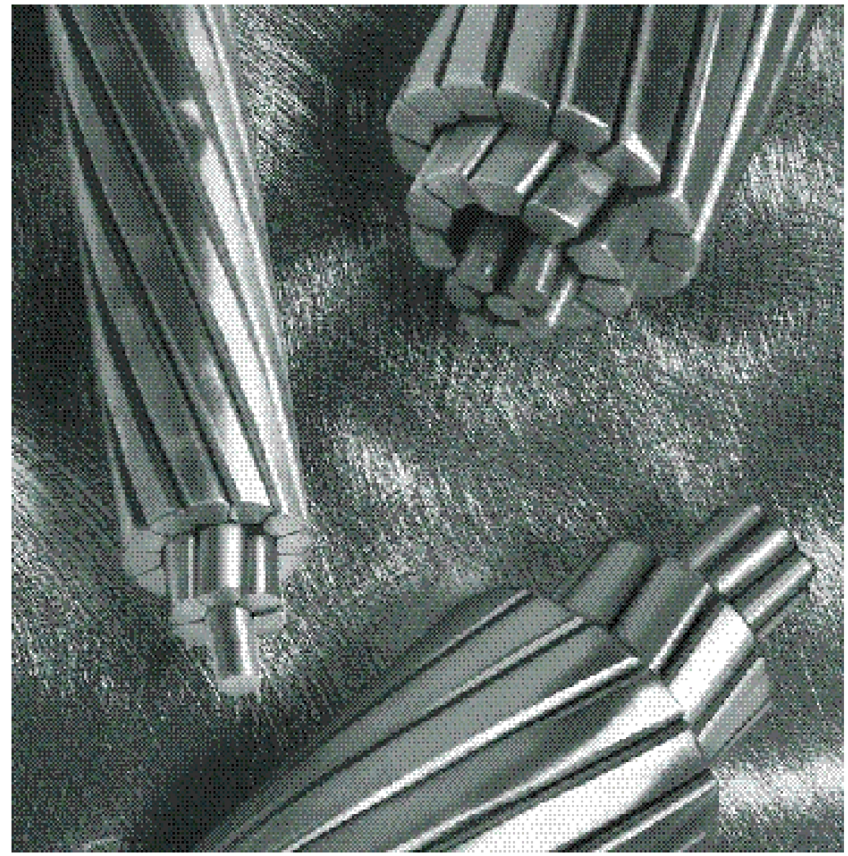
Transmission Lines

- Overhead Conductor
- Overhead Spacer Cable
- Underground Cable
- Three-Conductor Cable
- Service Cables

Overhead Conductors

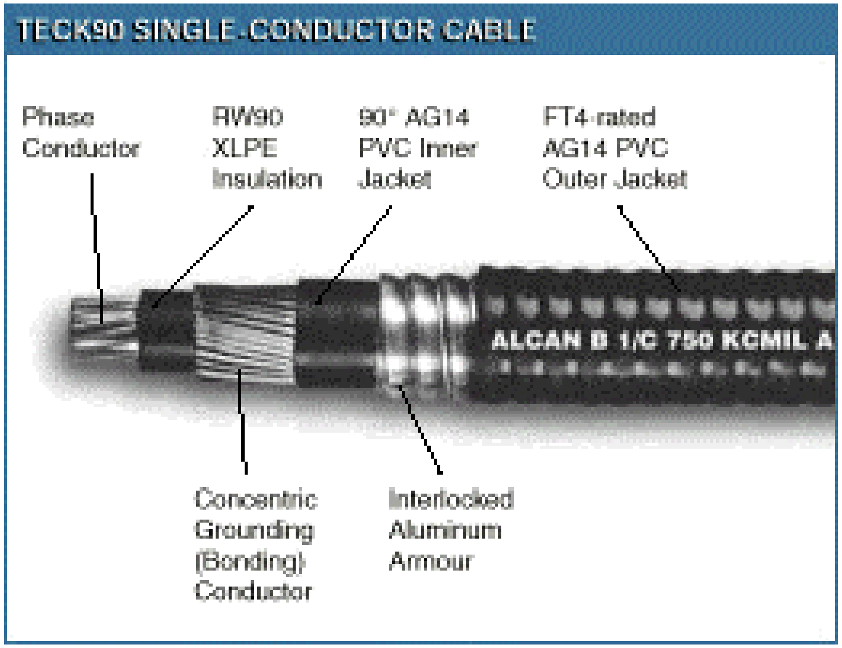
- **ACSR**
  Aluminum Conductor with inner Steel Reinforced strands
- **ACAR**
  Aluminum Conductor with inner Al allow Reinforced strands
- **ACSR/AW**
  Aluminum Conductor with inner Alumoweld Steel Reiforced strands
- **Aluminum - current carrying member**
- **Steel - structural support**

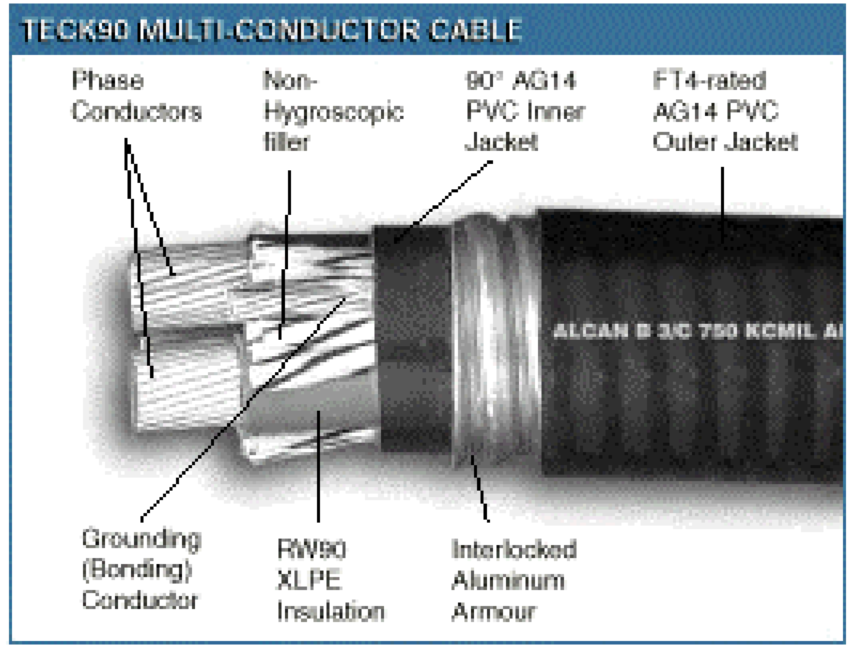


Overhead Cable

- Where conductor close proximity is required
- Insulating jacket surrounds each conductor
- Plastic spacers keep conductors from coming in contact with one another

Cables





Specification

CSA C22.2 No. 131 (TECK)
CSA C22.2 No. 174 (Hazardous Locations)
FT4-Rated: Vertical Flame Test - cable in cable tray (IEEE P1202)
AG14 inner and outer PVC jacket:
Maximum 14% acid gas emission by weight
FMRC Class 3972 Fire Test GP-2 (jacketed)
GP-1 (unjacketed)

Cables

- **Underground transmission and distribution cables**
- **Semiconducting material surrounds the conductor to grade the electric field**
- **Plastic jacket provides insulation and protection**
- **Neutral strands for an outer shell for protection and return currents**

Transmission Line Parameters

- **Line resistance**
  - dc resistance

$$R_{dc} = \frac{\rho\, l}{A}$$

ρ = conductor resistivity
l = conductor length
A = conductor cross-sectional area

  - ac resistance
    - skin effect
    - at 60 Hz:

$$R_{ac} = 1.02 \cdot R_{dc}$$

- **Temperature effects**
  - increased resistance at conductor temperature rises
  - wiring is rated for 65°C, 75°C, or 90°C
  - ambient temperature is 20°C

$$R_{new} = R_{old} \frac{T + t_{new}}{T + t_{old}}$$

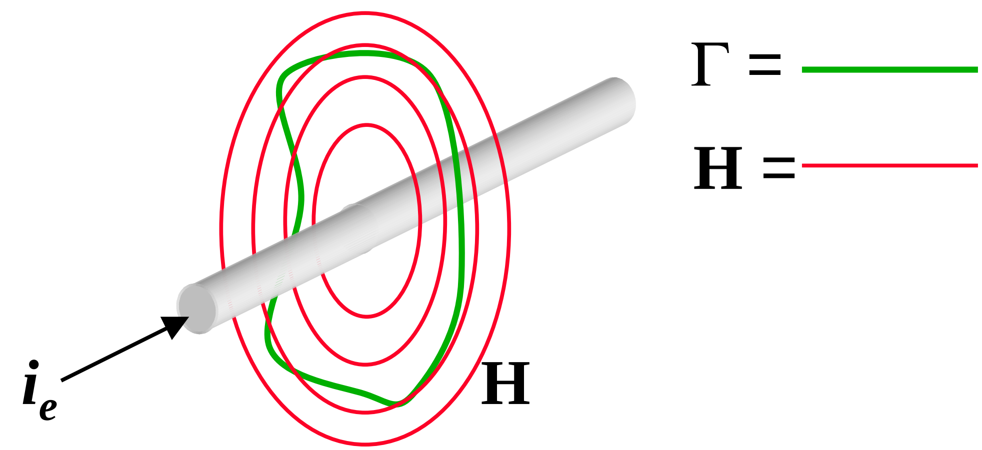
$$T_{Al} = 228°\mathrm{C}$$

Review of Magnetics and Inductance

Ampere's circuital law

$$F = \oint_{\Gamma} \mathbf{H} \cdot d\,\mathbf{l} = i_e$$

Integral of the scalar product of a closed path and the magnetic field equals the encircled current

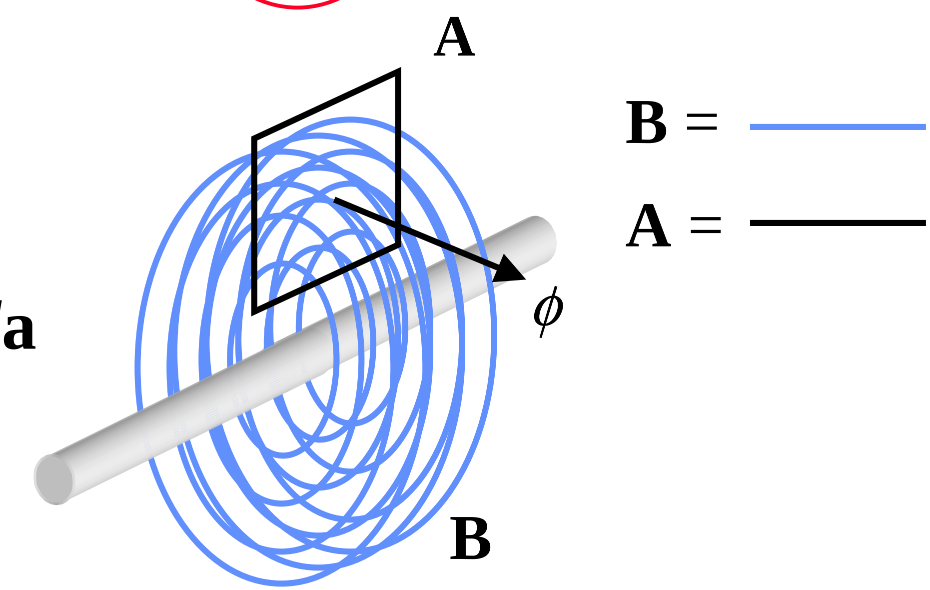


Magnetic Flux

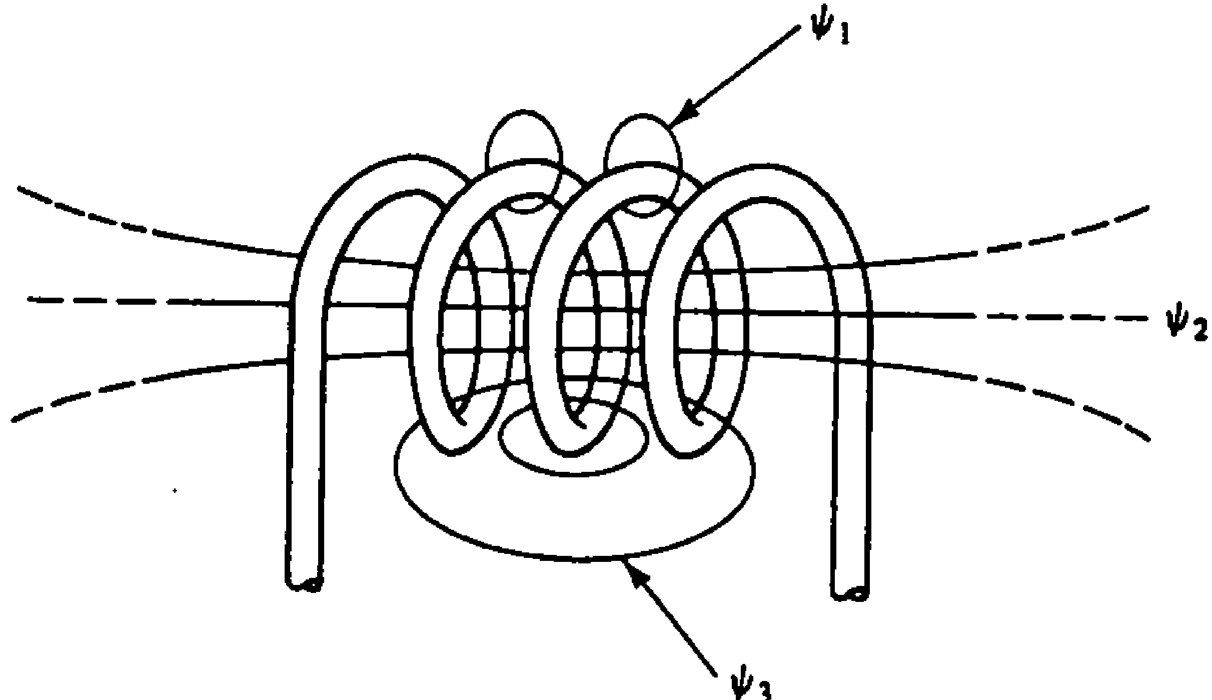
$$\mathbf{B} = \mu \mathbf{H}$$

Integral of the flux density that is normal to a defined area

$$\phi = \int_A \mathbf{B} \cdot d\mathbf{a}$$



Review of Magnetics and Inductance



Flux Linkage

$$\lambda = \sum_{i=1}^{N} \phi_i$$

Inductance

$$L = \frac{\lambda}{I}$$

$$= \frac{\sum \phi}{I} = \frac{\sum \int \mathbf{B} \cdot d\mathbf{a}}{I} = \frac{\sum \int \mu \mathbf{H} \cdot d\mathbf{a}}{I}$$

Inductance of a Single Conductor

- **Conditions:**
  - infinite straight wire is an approximation of a reasonably long wire
- **Assumptions:**
  - Image the wire to close at +/- infinity, establishing a kind of "one-turn coil" with the return path at infinity
  - Straight infinitely long wire of radius r
  - Uniform current density in the wire. Total current is I_x
  - Flux lines form concentric circles (i.e. $\mathbf{H}$ is tangential)
  - Angular symmetry - it suffices to consider H_x

Inductance of a Single Conductor

- **General:** $$\int_0^{2\pi x} H_x \cdot dl = I_x \quad \Rightarrow \quad H = \frac{I_x}{2\pi x}$$

- **Case 1: Points inside of the conductor ($x < r$)**

$$\frac{I}{\pi r^2} = \frac{I_x}{\pi x^2} \quad \rightarrow \quad H_x = \frac{I}{2\pi r^2} x \quad \rightarrow \quad B_x = \frac{\mu_0 I}{2\pi r^2} x$$

$$d\phi_x = B_x dx = \frac{\mu_0 I}{2\pi r^2} x dx \quad \rightarrow \quad d\lambda_x = \frac{x^2}{r^2} d\phi_x = \frac{\mu_0 I}{2\pi r^4} x^3 dx$$

$$\lambda_{int} = \int_0^r d\lambda_x = \frac{\mu_0 I}{2\pi r^4} \int_0^r x^3 dx = \frac{\mu_0 I}{8\pi} \quad \rightarrow \quad L_{int} = \frac{\mu_0}{8\pi} = 0.5 \times 10^{-7}$$

Inductance of a Single Conductor

- **Case 2: Points outside of the conductor ($x > r$)**

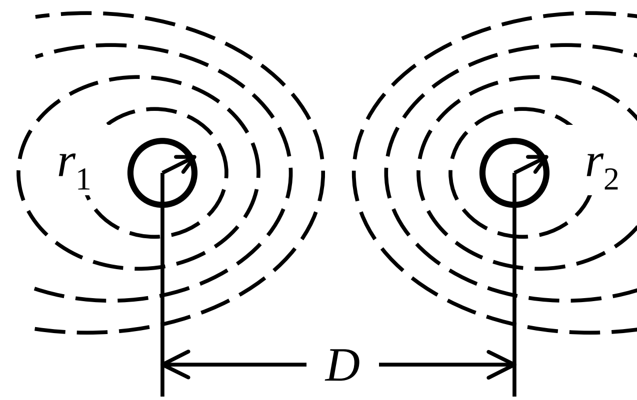
$$I_x = I \quad \rightarrow \quad B_x = \mu_0 H_x = \frac{\mu_0 I}{2\pi x}$$

$$d\phi_x = B_x dx = \frac{\mu_0 I}{2\pi x} dx \quad \rightarrow \quad d\lambda_x = d\phi_x = \frac{\mu_0 I}{2\pi x} dx$$

$$\lambda_{ext} = \int_{D_1}^{D_2} d\lambda_x = \frac{\mu_0 I}{2\pi} \int_{D_1}^{D_2} \frac{1}{x} dx = \frac{\mu_0 I}{2\pi} \ln \frac{D_2}{D_1} \quad \rightarrow \quad L_{ext} = 2\times 10^{-7} \ln \frac{D_2}{D_1}$$

Inductance of a Single-Phase Line

- **conductors of radii r_1 and r_2, separated by a distance D**



$$L_{1(ext)} = 2\times10^{-7}\ln\frac{D}{r_1}$$

$$L_1 = L_{1(int)} + L_{1(ext)} = 0.5\times10^{-7} + 2\times10^{-7}\ln\frac{D}{r_1}$$

$$L_1 = 2\times10^{-7}\left(\ln\frac{1}{r_1 e^{-1/4}} + \ln D\right)$$

$r_1 = r_2$ $\qquad$ $L_1 = L_2 = L$ $\qquad$ $r' = re^{-1/4} = D_S$

$$L = 2\times10^{-7}\left(\ln\frac{D}{re^{-1/4}}\right) = 2\times10^{-7}\left(\ln\frac{D}{r'}\right) = 2\times10^{-7}\left(\ln\frac{D}{D_S}\right)$$

Flux Linkage - Self and Mutual Inductances

From the 2 conductor case:

$$\begin{aligned} \lambda_1 &= L_{11} I_1 + L_{12} I_2 \\ \lambda_2 &= L_{21} I_1 + L_{22} I_2 \end{aligned} \qquad I_1 = -I_2 \rightarrow \qquad \begin{aligned} \lambda_1 &= L_{11} I_1 - L_{12} I_1 \\ \lambda_2 &= -L_{21} I_2 + L_{22} I_2 \end{aligned}$$

$$L_{11} = 2\times10^{-7}\left(\ln\frac{1}{r_1'}\right) \quad L_{22} = 2\times10^{-7}\left(\ln\frac{1}{r_2'}\right)$$

$$L_{12} = L_{21} = -2\times10^{-7}\left(\ln D\right) = 2\times10^{-7}\left(\ln\frac{1}{D}\right)$$

Total Inductance

General Case:

$$I_1 + I_2 + \cdots + I_i + \cdots + I_n = 0$$

$$\lambda_i = L_{ii} I_i + \sum_{j=1}^{n} L_{ij} I_j \qquad j \neq i$$

$$\lambda_i = 2 \times 10^{-7} \left(I_i \ln \frac{1}{r_i'} + \sum_{j=1}^{n} I_j \frac{1}{D_{ij}} \right) \qquad j \neq i$$

Inductance of Three-Phase Lines

- **Symmetrical spacing**

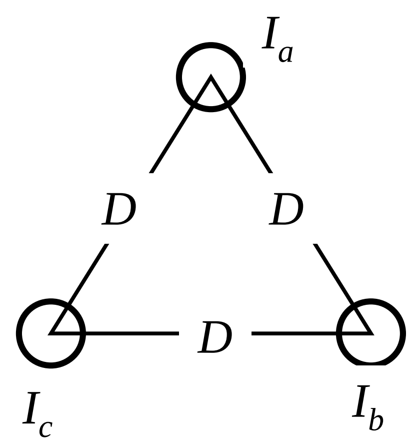
$$I_a + I_b + I_c = 0$$

$$\lambda_a = 2\times10^{-7}\left(I_a \ln\frac{1}{r'} + I_b \ln\frac{1}{D} + I_c \ln\frac{1}{D} \right)$$

$$\lambda_a = 2\times10^{-7}\left(I_a \ln\frac{1}{r'} - I_a \ln\frac{1}{D} \right)$$

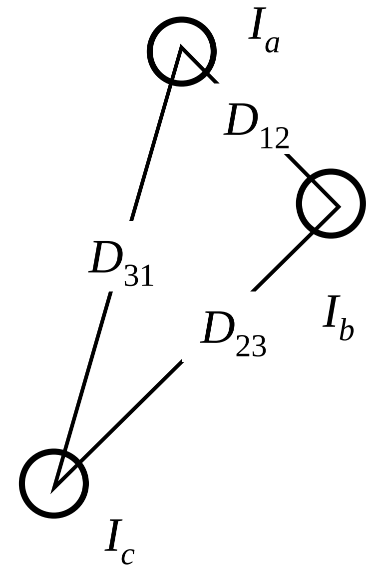
$$\lambda_a = 2\times10^{-7} I_a \ln\frac{D}{r'}$$

$$L = 0.2\ln\frac{D}{D_S}$$



Inductance of Three-Phase Lines

- **Asymmetrical spacing**



$$\lambda_a = 2\times10^{-7}\left(I_a \ln\frac{1}{r'} + I_b \ln\frac{1}{D_{12}} + I_c \ln\frac{1}{D_{13}}\right)$$

$$\lambda_b = 2\times10^{-7}\left(I_a \ln\frac{1}{r'} + I_b \ln\frac{1}{D_{21}} + I_c \ln\frac{1}{D_{23}}\right)$$

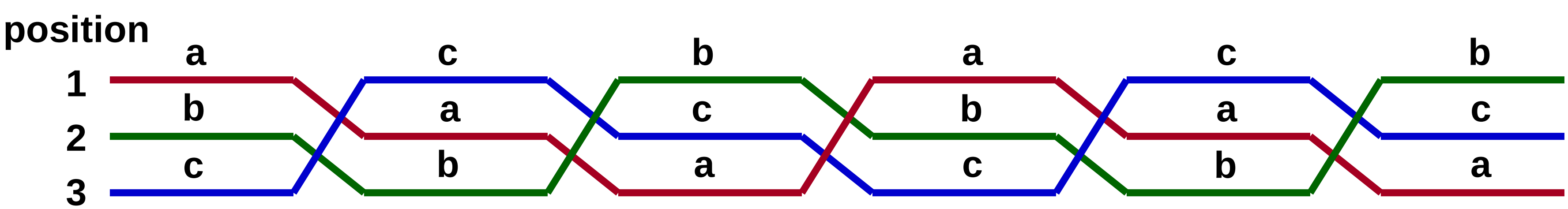
$$\lambda_c = 2\times10^{-7}\left(I_a \ln\frac{1}{r'} + I_b \ln\frac{1}{D_{31}} + I_c \ln\frac{1}{D_{32}}\right)$$

$$\boldsymbol{\lambda} = \mathbf{LI}$$

$$L = 2\times10^{-7}\begin{bmatrix} \ln\frac{1}{r'} & \ln\frac{1}{D_{12}} & \ln\frac{1}{D_{13}} \\ \ln\frac{1}{D_{21}} & \ln\frac{1}{r'} & \ln\frac{1}{D_{23}} \\ \frac{1}{D_{31}} & \frac{1}{D_{32}} & \ln\frac{1}{r'} \end{bmatrix}$$

Transposition

- **The practice of equilateral arrangement of phases is not convenient**
  - horizontal or vertical configurations are most popular
  - Symmetry is lost - unbalanced conditions
- **restore balanced conditions by the method of transposition of lines**
  - Average inductance of each phase will be the same



  - Each phase occupies each position for the same fraction of the total length of the line

Review of Electric Fields

Gauss's law

$$q_e = \int_A \mathbf{D} \cdot d\mathbf{a}$$

Electric field

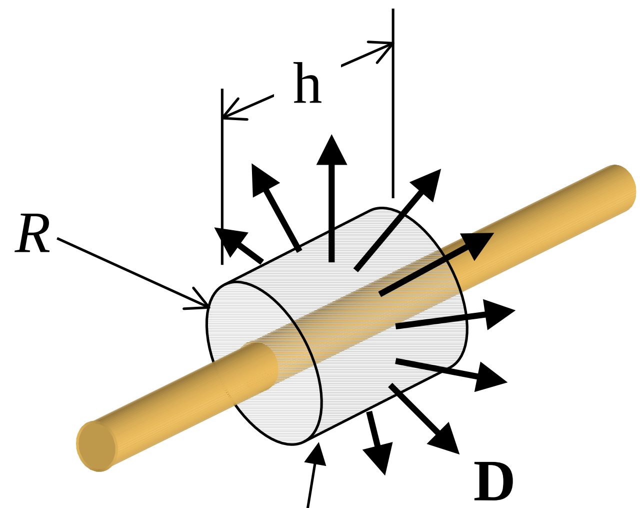
$$\mathbf{D} = \varepsilon \mathbf{E}$$

Electric field

$$v_{12} = v_{D_1} - v_{D_2} = -\int_{D_1}^{D_2} \mathbf{E} \cdot d\mathbf{l}$$

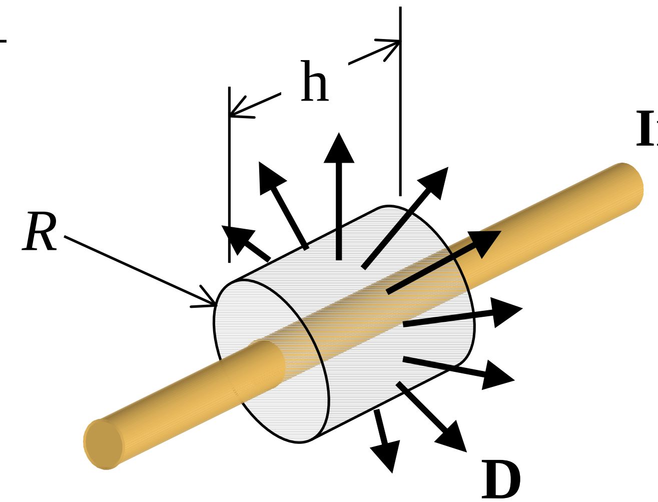
Capacitance

$$q = C\,v$$



Infinite Straight Wire

$$v_{12} = \int_{D_1}^{D_2} \frac{q}{2\pi\varepsilon_0 x} dx = \frac{q}{2\pi\varepsilon_0} \ln\frac{D_2}{D_1}$$



$$v_{12(q1)} = \frac{q_1}{2\pi\varepsilon_0} \ln\frac{D}{r}$$

$$v_{21(q2)} = \frac{q_2}{2\pi\varepsilon_0} \ln\frac{r}{D}$$

$$C = \frac{q}{v}$$

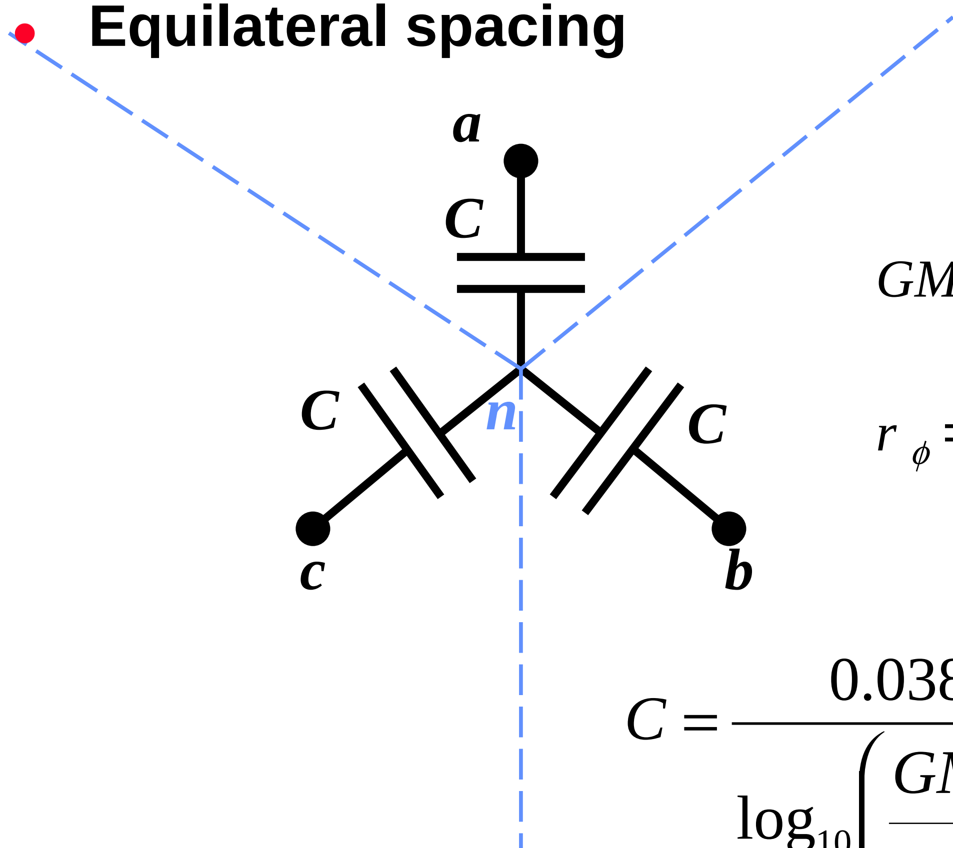
$$v_{12} = v_{12(q1)} + v_{21(q2)} = \frac{q_1}{2\pi\varepsilon_0} \ln\frac{D}{r} + \frac{q_2}{2\pi\varepsilon_0} \ln\frac{r}{D}$$

$$C = \frac{2\pi\varepsilon}{\ln\frac{D}{r}}$$

$$v_{12} = \frac{q}{\pi\varepsilon_0} \ln\frac{D}{r}$$

Three-Phase Capacitance

- **Equilateral spacing**



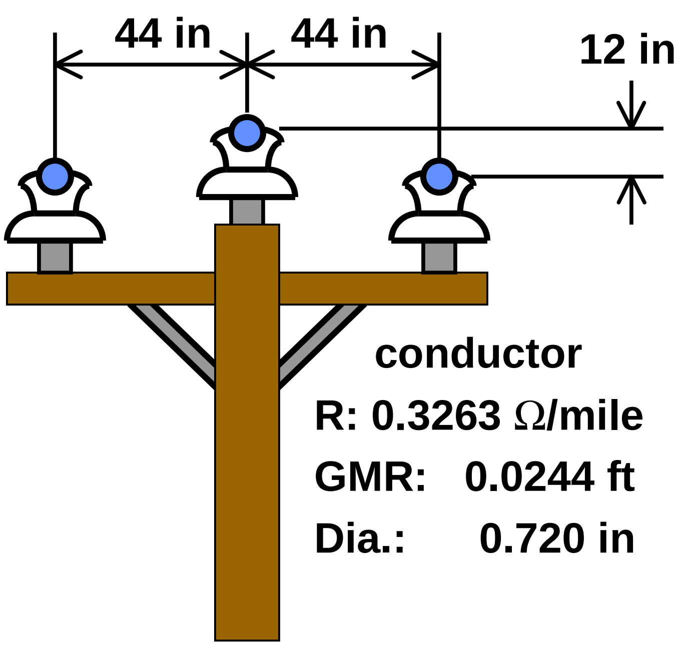
$GMD_\phi =$ geometeric mean distance between conductors

$r_\phi =$ conductor radius

$$C = \frac{0.0389}{\log_{10}\left(\frac{GMD_\phi}{r_\phi}\right)} \quad \mu\text{F per mile per phase}$$

Example

Calculate the resistance, inductive reactance, and capacitive reactance per phase and rated current carrying capacity for the overhead line shown. Assume the line operates at 60 Hz



$$GMD_\phi = \sqrt[3]{d_{12}\ d_{23}\ d_{13}} = \sqrt[3]{(45.6)(88)(45.6)}$$

$$= 56.8\text{ in} = 4.73\text{ ft}$$

$$Z_a = (0.3263) + j\ 0.2794\frac{(60)}{60}\log_{10}\left(\frac{4.73}{0.0244}\right)$$

$$= 0.326 + j0.639 \quad \Omega/\text{mi}$$

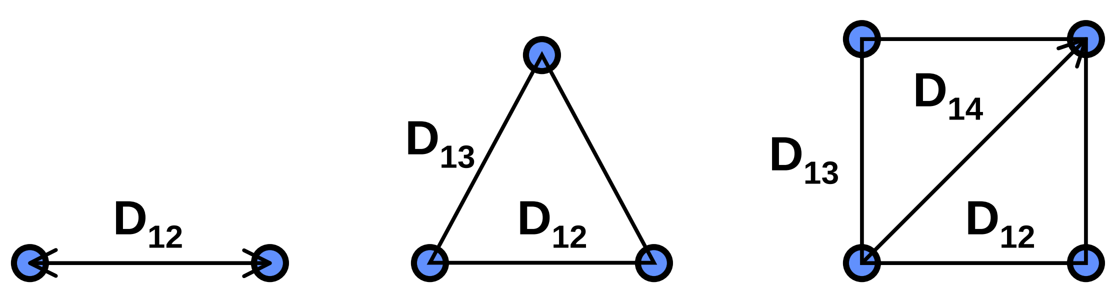
$$r_\phi = \tfrac{1}{2}dia = \tfrac{1}{2}(0.720\text{ in})\cdot\tfrac{1}{12} = 0.03\text{ ft}$$

$$C = \frac{0.0389}{\log_{10}(4.73/0.03)} = 0.177\ \mu\text{F/mi/phs}$$

$$X_C = 1/(2\pi\ 60\cdot 0.177\ \mu\text{F}) = 149.9 \quad \Omega\,\text{mi}$$

Conductor Bundling

- Commonly used to reduce the electric field strength at the conductor surface
- Used on overhead lines above 230 kV
- Conductors are connected in parallel
- Typical bundled conductor configurations



2 conductors 3 conductors 4 conductors

Conductor Bundling

- **The use of bundled conductors effects the impedance of the line, the GMR_ϕ, the GMD_ϕ, and the equivalent radius**
- **GMD_ϕ: the distance between the center of each bundle is used**
- **GMR_ϕ:**

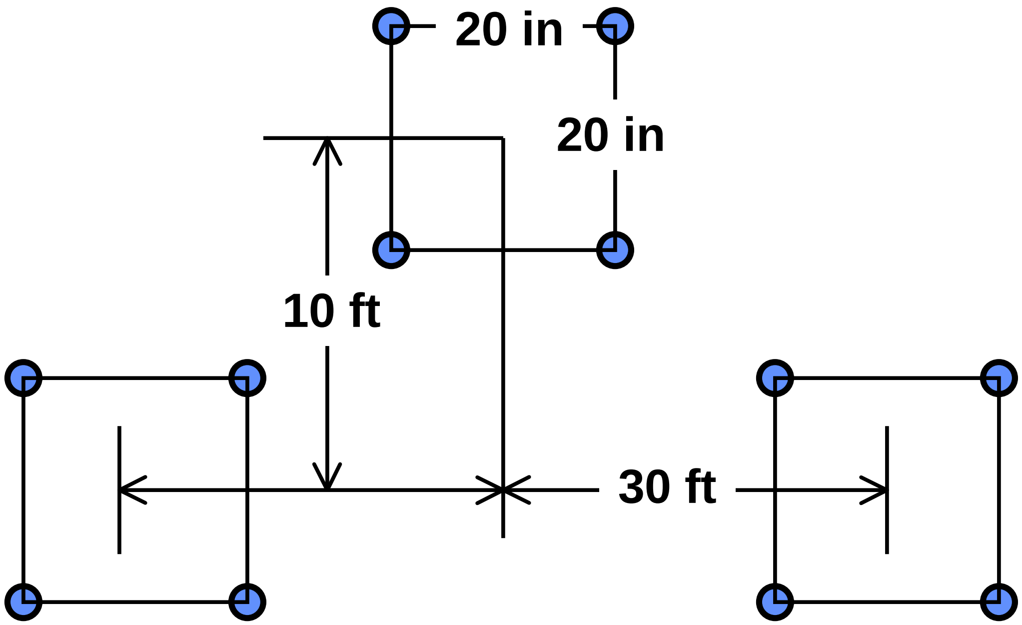
$$GMR'_\phi = \sqrt[n]{GMR_\phi \cdot \prod_{i=2}^{n} D_{1i}}$$

- **Equivalent radius**

$$r'_\phi = \sqrt[n]{r_\phi \cdot \prod_{i=2}^{n} D_{1i}}$$

Example

Calculate the resistance, inductive reactance, and capacitive reactance of the overhead line shown. Assume the line operates at 60 Hz



conductor

R: 0.1204 Ω/mile

GMR: 0.0403 ft

Dia.: 1.196 in

Example

$$R' = \tfrac{1}{4} \cdot 0.1204 = 0.0301 \quad \Omega / \text{mi}$$

$$GMD_\phi = \sqrt[3]{D_{12}\ D_{23}\ D_{13}} = \sqrt[3]{(31.6)(60)(31.6)} = 39.15\ \text{ft}$$

$$GMR_\phi = \sqrt[4]{(0.0403)(1.67)(1.414)(1.67)} = 0.7178\ \text{ft}$$

$$Z_a = (0.0301) + j\ 0.2794 \frac{(60)}{60} \log_{10}\left(\frac{39.15}{0.7178}\right)$$

$$= 0.0301 + j0.485 \quad \Omega / \text{mi}$$

$$r_\phi = \tfrac{1}{2}\, dia = \tfrac{1}{2}(1.196\ \text{in}) \cdot \tfrac{1}{12} = 0.0498\ \text{ft}$$

$$r'_\phi = \sqrt[4]{(0.0498)(1.67)(1.414)(1.67)} = 0.7568\ \text{ft}$$

$$C = \frac{0.0389}{\log_{10}(39.15/0.7568)} = 0.0227\ \mu\text{F/mi/phs}$$

$$\mathrm{X_C} = 1/(2\pi\ 60 \cdot 0.177\ \mu\text{F}) = 116.85 \quad \Omega\,\text{mi}$$

Transmission Line Modeling

- **Transmission lines are represented by an equivalent circuit with parameters on a per-phase basis**
  - Voltages are expressed as phase-to-neutral
  - Currents are expressed for one phase
  - The three phase system is reduced to an equivalent single-phase
- **All lines are made up of distributed series inductance and resistance, and shunt capacitance and conductance**
  - Line parameters: R, L, C, & G
- **Three types of models**
  - depend on the length and the voltage level
  - short, medium, and long length line models

ABCD Two-Port Network

- **All transmission line models may be described as a two-port network**
- **The ABCD two-port network is the most common representation**
- **The network is described by the four constants:** A, B, C, **&** D
- **Network equations:**
  - circuit equations

$$V_S = A\,V_R + B\,I_R$$

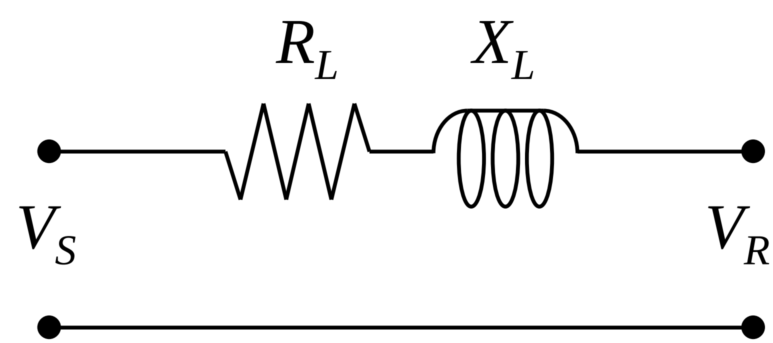
$$I_S = C\,V_R + D\,I_R$$

  - matrix form

$$\begin{bmatrix} V_S \\ I_S \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_R \\ I_R \end{bmatrix}$$

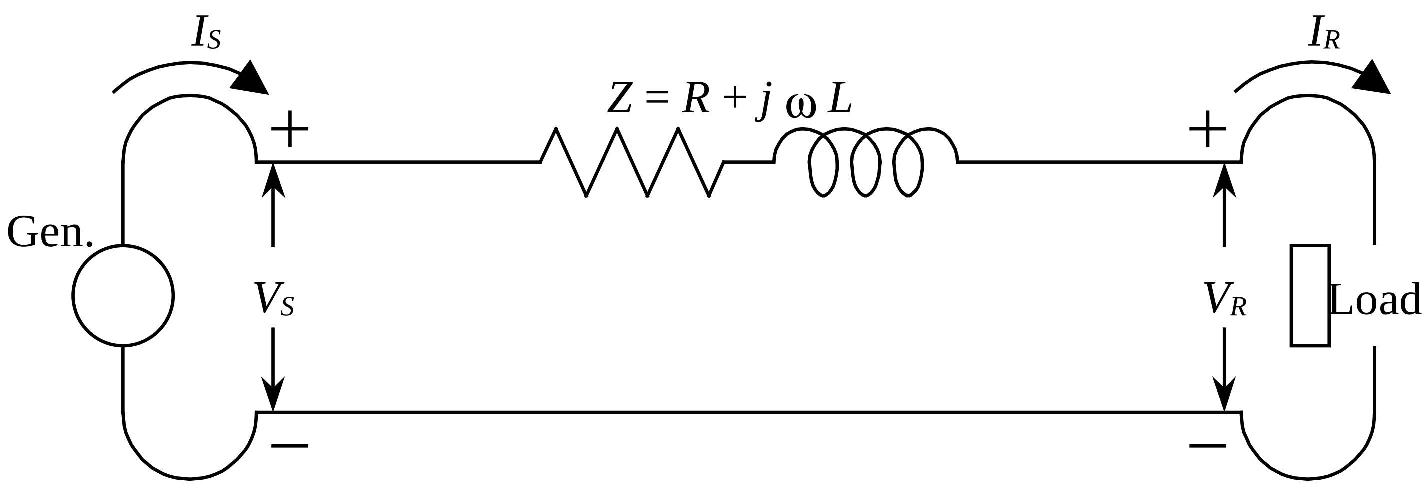
Short Transmission Line Model

- **The short transmission line model may be used when**
  - The line length is less than 50 miles (80 km), or
  - The line voltage is not over 69 kV
- **Modeling of the transmission line parameters**
  - The shunt capacitance and conductance are ignored
  - The line resistance and reactance are treated as lumped parameters
- **Circuit of the short model**



Short Transmission Line Model

- **Circuit analysis of the short line model**



$$
\begin{aligned}
I_S &= I_R \\
V_S &= V_R + I_R\,(R + j\,\omega_s\,L) \\
&= V_R + I_R\,Z
\end{aligned}
$$

Two-Port Representation

- **Circuit Equations:**

$$V_S = V_R + Z_{line}\, I_R$$

$$I_S = I_R$$

  - Matrix representation:

$$\begin{bmatrix} V_S \\ I_S \end{bmatrix} = \begin{bmatrix} 1 & Z_{line} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} V_R \\ I_R \end{bmatrix}$$

  - ABCD values:

$$A = 1$$

$$B = Z_{line}$$

$$C = 0$$

$$D = A = 1$$

Short Transmission Line Example

- **40 km, 220 kV transmission line has per phase**
  - R = 0.15 Ω/km L = 1.3263 mH/km
- **Find V, S, V.R., and η at the sending end of the line for**
  - 381 MVA load at 0.8 lagging pf at 220 kV

$$Z = (r + j\omega L)\ell = \left(0.15\,\Omega + j\,2\pi \times 60 \times 1.3263 \times 10^{-3}\right) \cdot 40$$

$$Z = 6 + j20\,\Omega$$

$$V_R = \frac{220{,}000\angle 0^\circ}{\sqrt{3}} = 127{,}000\angle 0^\circ$$

$$S_{R(3\phi)} = 381\angle\left(\cos^{-1} 0.8\right) = 381\angle 36.9^\circ = 304.8 + j228.6\,\text{MVA}$$

Short Transmission Line Example

$$I_R = \frac{S^*_{R(3\phi)}}{3\,V^*_R} = \frac{381\times10^6\angle -36.9^\circ}{3\times127{,}000\angle 0^\circ} = 1000\angle -36.9^\circ \text{ A}$$

$$V_S = V_R + Z\,I_R = 127{,}000\angle 0^\circ + (6 + j20)(1000\angle -36.9^\circ)$$

$$= 144{,}330\angle 4.93^\circ$$

$$\left|V_{S-LL}\right| = \sqrt{3}\cdot\left|V_S\right| = 250\text{ kV}$$

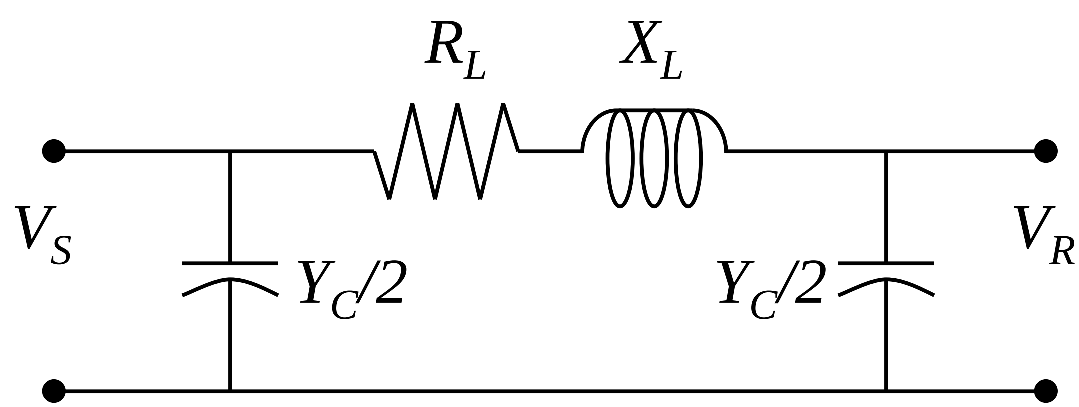
$$S_{S(3\phi)} = 3\cdot V_S\cdot I_S^* = 3\cdot(144{,}330\angle 4.93^\circ)(1000\angle -36.9^\circ)$$

$$= 322.8 + j288.6 = 433\angle 41.8^\circ\text{ MVA}$$

$$\text{VR\%} = \frac{250 - 220}{220}\times100\% = 13.6\% \qquad \eta = \frac{304.8}{322.8}\times100\% = 94.4\%$$

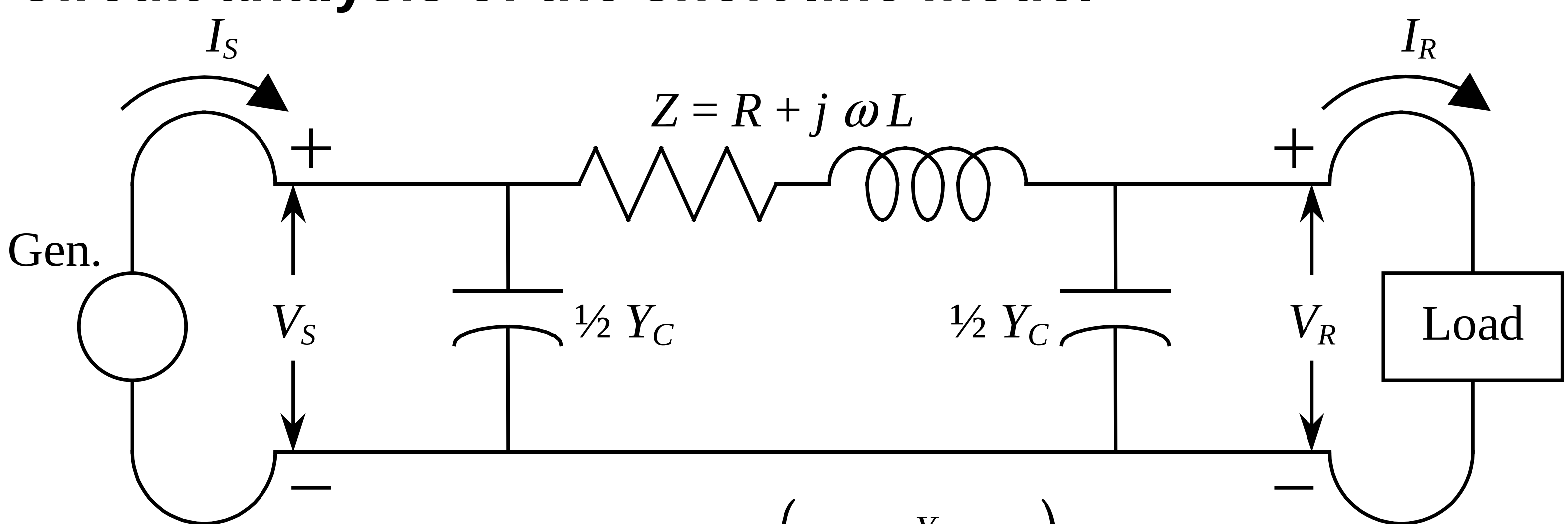
Medium Transmission Line Model

- **The medium transmission line model may be used when**
  - The line length is greater than 50 miles (80 km)
  - The line length is less than 150 miles (250 km)
- **Modeling of the transmission line parameters**
  - Half of the shunt capacitance is considered to be lumped at each end of the line
  - The line resistance and reactance are treated as lumped parameters
- **Circuit model:**



Medium Transmission Line Model

- **Circuit analysis of the short line model**



$$
\begin{aligned}
V_S &= V_R + Z_{line}\left(I_R + \tfrac{Y_C}{2} V_R\right) \\
&= \left(1 + \frac{Z_{line}\,Y_C}{2}\right) V_R + Z_{line} I_R \\
I_S &= \left(I_R + \tfrac{Y_C}{2} V_R\right) + \tfrac{Y_C}{2} V_S \\
&= Y_C\left(1 + \frac{Z_{line}\,Y_C}{4}\right) V_R + \left(1 + \frac{Z_{line}\,Y_C}{2}\right) I_R
\end{aligned}
$$

Two-Port Representation

- **Circuit Equations:**

$$V_S = V_R + Z_{line}\left(I_R + \frac{Y_C}{2}V_R\right)$$
$$= \left(1 + \frac{Z_{line}Y_C}{2}\right)V_R + Z_{line}I_R$$
$$I_S = \left(I_R + \frac{Y_C}{2}V_R\right) + \frac{Y_C}{2}V_S$$
$$= Y_C\left(1 + \frac{Z_{line}Y_C}{4}\right)V_R + \left(1 + \frac{Z_{line}Y_C}{2}\right)I_R$$

  - Matrix representation:

$$\begin{bmatrix} V_S \\ I_S \end{bmatrix} = \begin{bmatrix} 1 + \frac{Z_{line}Y_C}{2} & Z_{line} \\ Y_C\left(1 + \frac{Z_{line}Y_C}{4}\right) & 1 + \frac{Z_{line}Y_C}{2} \end{bmatrix} \begin{bmatrix} V_R \\ I_R \end{bmatrix}$$

  - ABCD values:

$$A = 1 + \frac{Z_{line}Y_C}{2} \qquad B = Z_{line}$$
$$C = Y_C\left(1 + \frac{Z_{line}Y_C}{4}\right) \qquad D = 1 + \frac{Z_{line}Y_C}{2}$$

Medium Transmission Line Example

- **130 km, 345 kV transmission line has per phase**
  - R = 0.036 Ω/km L = 0.80 mH/km C = 0.0112 uF/km
- **Find V and S at the sending end of the line for**
  - 270 MVA load at 0.8 lagging pf at 325 kV

$$Z = (r + j\omega L)\ell = \left(0.036\,\Omega + j\,2\pi \times 60 \times 0.8 \times 10^{-3}\right) \cdot 130$$

$$= 4.68 + j39.2\,\Omega$$

$$Y = (j\omega C)\ell = \left(j\,2\pi \times 60 \times 0.0112 \times 10^{-6}\right) \cdot 130 = j0.549 \text{ siemens}$$

$$V_R = \frac{325{,}000\angle 0^\circ}{\sqrt{3}} = 187{,}600\angle 0^\circ$$

$$S_{R(3\phi)} = 270\angle\left(\cos^{-1} 0.8\right) = 270\angle 36.9^\circ = 216 + j162\,\text{MVA}$$

Medium Transmission Line Example

$$I_R = \frac{S^*_{R(3\phi)}}{3\,V^*_R} = \frac{270\times 10^6\angle -36.9^\circ}{3\times 187{,}600\angle 0^\circ} = 480\angle -36.9^\circ\ \mathrm{A}$$

$$ABCD = \begin{bmatrix} 0.989 + j0.001284 & 4.68 + j39.2 \\ -3.53\times 10^{-7} + j5.46\times 10^{-4} & 0.989 + j0.001284 \end{bmatrix}$$

$$V_S = A\,V_R + B\,I_R = (187{,}600\angle 0^\circ)(0.989 + j0.001284) + (480\angle -36.9^\circ)(4.68 + j39.2)$$

$$= 199{,}160\angle 4.02^\circ$$

$$I_S = C\,V_R + D\,I_R = (187{,}600\angle 0^\circ)(-3.53\times 10^{-7} + j5.46\times 10^{-4}) + (480\angle -36.9^\circ)(0.989 + j0.001284)$$

$$= 421.5\angle -25.58^\circ$$

Medium Transmission Line Example

$$\left|V_{S-LL}\right| = \sqrt{3} \cdot \left|V_S\right| = 345\ \mathrm{kV}$$

$$S_{S(3\phi)} = 3 \cdot V_S \cdot I_S^* = 3 \cdot (199{,}160\angle 4.02^\circ)(421\angle -25.58^\circ)$$

$$= 218.9 + j124.2\ \mathrm{MVA} \quad \mathrm{pf} = 0.87$$

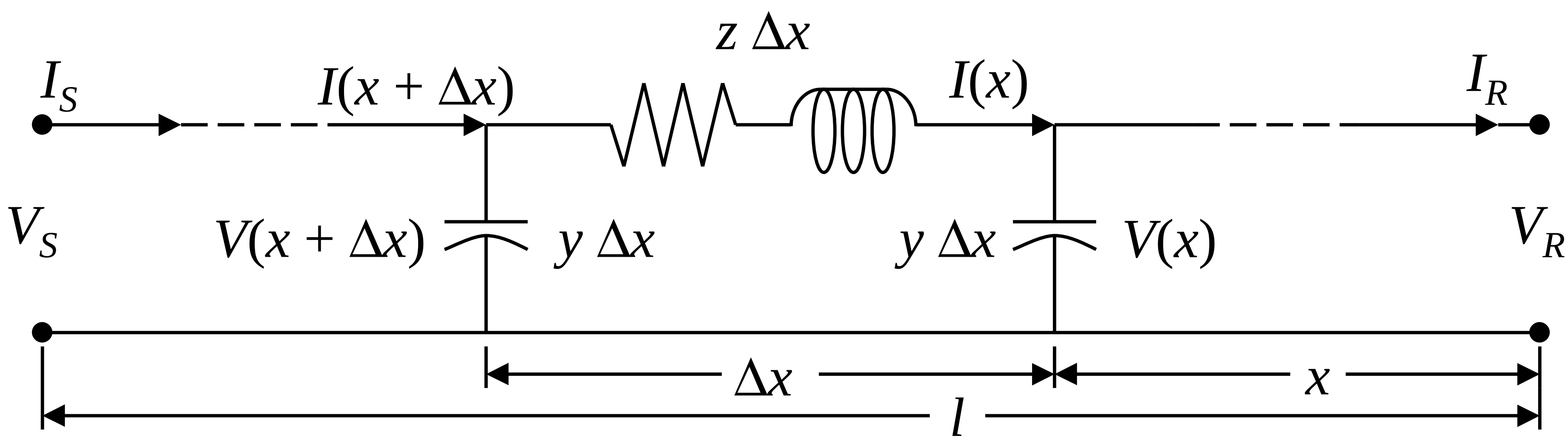
$$\mathrm{VR\%} = \frac{V_{R(NL)} - V_{R(FL)}}{V_{R(FL)}} \times 100\% = \frac{V_{S(FL)} / A - V_{R(FL)}}{V_{R(FL)}} \times 100\%$$

$$= \frac{345 / \left|0.989 + j0.001284\right| \text{-} 325}{325} \times 100\% = 7.3\%$$

Long Transmission Line Model

- **The long transmission line model are used when**
  - The line length is greater than 150 miles (250 km)
- **Modeling of the transmission line parameters**
  - Accuracy obtained by using distributed parameters
  - The series impedance per unit length is z
  - The shunt admittance per unit length is y

Long Transmission Line Model



$$V(x+\Delta x) = V(x) + z\,\Delta x\,I(x)$$

$$\frac{V(x+\Delta x) - V(x)}{\Delta x} = z\,I(x)$$

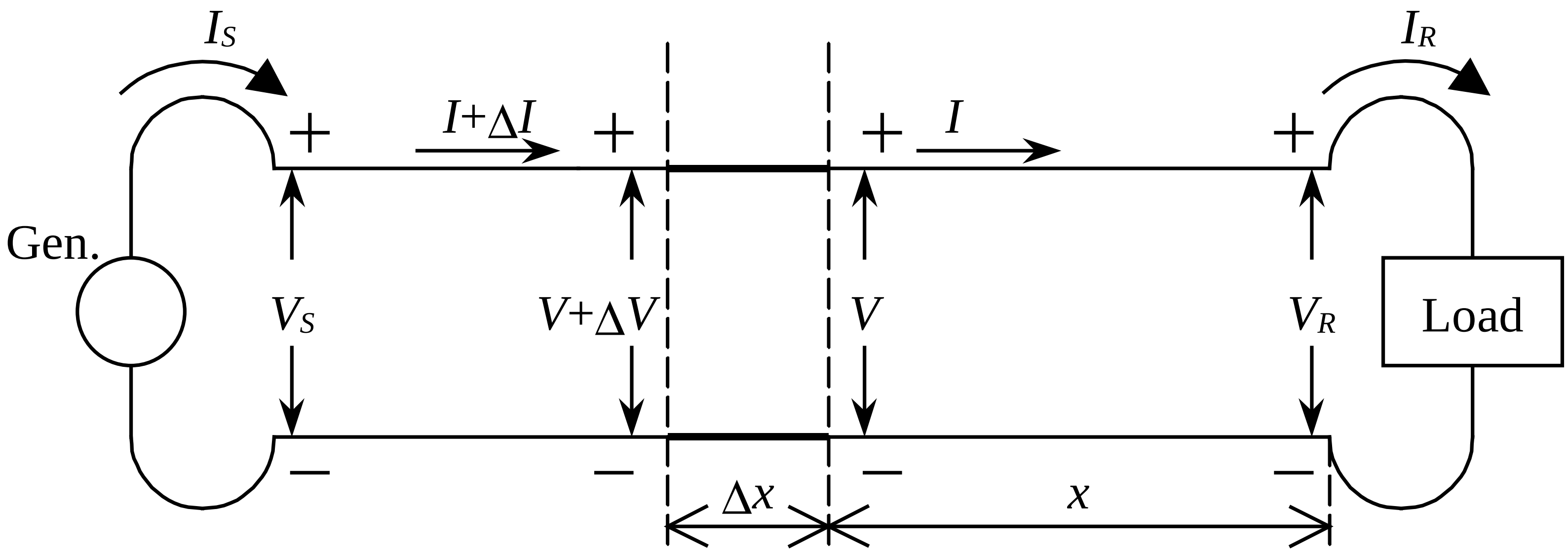
$$\text{limit as } \Delta x \to 0 \quad \frac{dV(x)}{dx} = z\,I(x)$$

$$I(x+\Delta x) = I(x) + y\,\Delta x\,V(x+\Delta x)$$

$$\frac{I(x+\Delta x) - I(x)}{\Delta x} = y\,V(x+\Delta x)$$

$$\text{limit as } \Delta x \to 0 \quad \frac{dI(x)}{dx} = y\,V(x)$$

Long Transmission Line Model



$$\frac{d^2V(x)}{dx^2} = z\,\frac{dI(x)}{dx} \qquad \frac{d^2I(x)}{dx^2} = y\,\frac{dV(x)}{dx}$$

$$\frac{d^2V(x)}{dx^2} = z\,\big(y\,V(x)\big) \qquad \frac{d^2I(x)}{dx^2} = y\,\big(z\,I(x)\big)$$

$\gamma^2 = z\,y$ propagation constant

Long Transmission Line Model

$$\frac{d^2V(x)}{dx^2} = \gamma^2 V(x)$$

$$V = A_1\, e^{\gamma x} + A_2\, e^{-\gamma x}$$

$$\gamma = \alpha + j\beta = \sqrt{z\, y} = \sqrt{(r + j\omega L)(g + j\omega C)}$$

$$I(x) = \frac{1}{z}\frac{dV(x)}{dx} = \frac{\gamma}{z}\left(A_1\, e^{\gamma x} - A_2\, e^{-\gamma x}\right) = \sqrt{\tfrac{y}{z}}\left(A_1\, e^{\gamma x} - A_2\, e^{-\gamma x}\right)$$

$$Z_c = \sqrt{z/y} \qquad I(x) = \frac{1}{Z_c}\left(A_1\, e^{\gamma x} - A_2\, e^{-\gamma x}\right)$$ characteristic impedance

$$@\, x = 0 \Rightarrow \quad A_1 = \frac{V_R + I_R\, Z_c}{2} \qquad A_2 = \frac{V_R - I_R\, Z_c}{2}$$

Long Transmission Line Model

$$V(x) = \frac{V_R + Z_c I_R}{2} e^{x\sqrt{y\,z}} + \frac{V_R - Z_c I_R}{2} e^{-x\sqrt{y\,z}}$$

$$I(x) = \frac{V_R/Z_c + I_R}{2} e^{x\sqrt{y\,z}} - \frac{V_R/Z_c - I_R}{2} e^{-x\sqrt{y\,z}}$$

$$V(x) = \frac{e^{x\sqrt{y\,z}} + e^{-x\sqrt{y\,z}}}{2} V_R + Z_c \frac{e^{x\sqrt{y\,z}} - e^{-x\sqrt{y\,z}}}{2} I_R$$

$$I(x) = \frac{1}{Z_c} \frac{e^{x\sqrt{y\,z}} - e^{-x\sqrt{y\,z}}}{2} V_R + \frac{e^{x\sqrt{y\,z}} + e^{-x\sqrt{y\,z}}}{2} I_R$$

$$V(x) = \cosh\left(x\sqrt{y\,z}\right) V_R + Z_c \sinh\left(x\sqrt{y\,z}\right) I_R$$

$$I(x) = \frac{1}{Z_c} \sinh\left(x\sqrt{y\,z}\right) V_R + \cosh\left(x\sqrt{y\,z}\right) I_R$$

Hyperbolic Functions

$$\sinh\theta = \frac{e^{\theta} - e^{-\theta}}{2}$$

$$\cosh\theta = \frac{e^{\theta} + e^{-\theta}}{2}$$

Two-Port Representation

$$let \quad x \rightarrow \ell$$

$$V_S = \cosh(\gamma \ell) V_R + Z_c \sinh(\gamma \ell) I_R$$

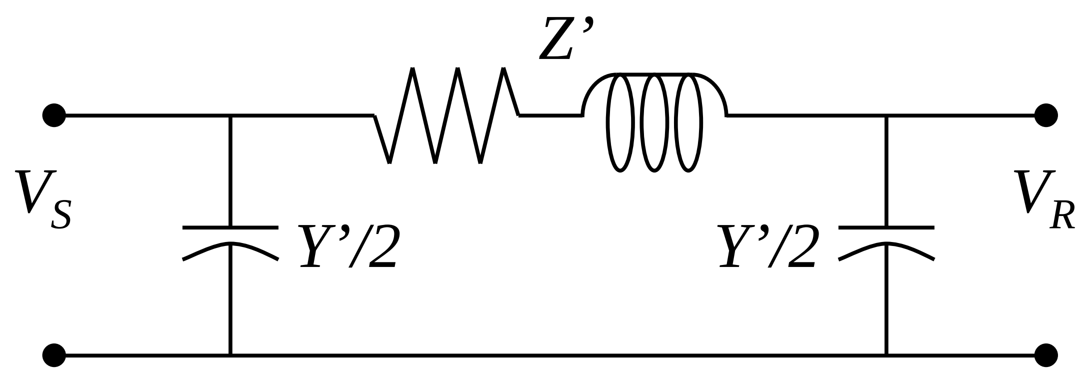
$$I_S = \frac{1}{Z_c} \sinh(\gamma \ell) V_R + \cosh(\gamma \ell) I_R$$

$$ABCD = \begin{bmatrix} \cosh(\gamma \ell) & Z_c \sinh(\gamma \ell) \\ \frac{1}{Z_c} \sinh(\gamma \ell) & \cosh(\gamma \ell) \end{bmatrix}$$

$$\gamma = \sqrt{z\, y} \qquad Z_c = \sqrt{\frac{z}{y}}$$

Pi-Model of a Long Transmission Line

- **Represent a long transmission line as a pi-model for circuit analysis**
- **The circuit:**



- **Find the values for *Z'* and *Y'***

$$V_S = \left(1+\frac{Z'Y'}{2}\right)V_R + Z'I_R \qquad \rightarrow Z' = Z_c \sinh(\gamma\,\ell)$$

$$I_S = Y'\left(1+\frac{Z'Y'}{4}\right)V_R + \left(1+\frac{Z'Y'}{2}\right)I_R \qquad \rightarrow$$

$$\frac{Y'}{2} = \frac{1}{Z'}\left(\cosh(\gamma\,\ell)-1\right) = \frac{\cosh(\gamma\,\ell)-1}{Z_c \sinh(\gamma\,\ell)} = \frac{1}{Z_c}\tanh\left(\frac{\gamma\,\ell}{2}\right)$$

Long Transmission Line Example

- **250 km, 500 kV transmission line has per phase**
  - z = 0.045 + j 0.4 Ω/km   Y = j 4. 0 uS/km
- **Find ABCD for a pi model of the long transmission line**

$$Z_c = \sqrt{\frac{z}{y}} = \sqrt{\frac{0.045 + j0.4}{4\times 10^{-6}}} = 316.7 \text{ - j}17.76$$

$$\gamma = \sqrt{zy} = \sqrt{(0.045 + j0.4)(4\times 10^{-6})} = 7.104\times 10^{-5} + j0.001267$$

$$Z' = Z_c \sinh(\gamma\,\ell) = 10.88 + j98.36$$

$$\frac{Y'}{2} = \frac{1}{Z_c}\tanh\left(\frac{\gamma\,\ell}{2}\right) = j0.001008$$

Long Transmission Line Example

$$Z' = 10.88 + j98.36$$

$$\frac{Y'}{2} = j0.001008$$

$$A = D = \left(1 + \tfrac{Z'Y'}{2}\right) = 0.9504 + j0.0055$$

$$B = Z' = 10.88 + j98.36$$

$$C = Y'\left(1 + \tfrac{Z'Y'}{4}\right) = j0.00100$$